

11.13 The Snowplough Problem

WORKSHEET

The question

Two identical snowploughs plough the same stretch of road in the same direction. The first starts at time $t = 0$ when the depth of snow is h metres, and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of $k \text{ m s}^{-1}$. It may be assumed that each snowplough moves at a speed equal to $k/(\alpha z) \text{ m s}^{-1}$ with α a constant, where z is the depth of snow it is ploughing, and that it clears all the snow.

- (i) Show that the time taken for the first snowplough to travel x metres is

$$(e^{\alpha x} - 1) \frac{h}{k} \text{ seconds.}$$

- (ii) Show that at time $t > T$, the second snowplough has moved y metres, where t satisfies

$$\frac{1}{\alpha} \frac{dt}{dy} = t - (e^{\alpha y} - 1) \frac{h}{k}.$$

Verify that the required solution of this equation is

$$t = (e^{\alpha y} - 1) \frac{h}{k} + \left(T - \frac{\alpha hy}{k}\right) e^{\alpha y}$$

and deduce that the snowploughs collide when they have moved a distance $kT/(\alpha h)$ metres.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The first plough meets snow of depth $h + kt$ at time t . What is its speed? Is it easier to find x as a function of t , or t as a function of x ?

STAGE 2 The second plough's equation



Ask yourself The second plough reaches position y at time t . How deep is the snow there? (What was the depth just after the first plough passed?)

STAGE 3 Solving



Ask yourself This is linear in t : $\frac{dt}{dy} - \alpha t = -\frac{\alpha h}{k}(e^{\alpha y} - 1)$. What integrating factor works? What is the starting condition?

STAGE 4 The collision



Ask yourself The ploughs collide when both reach the same point at the same time. Compare $t_2(y)$ with $t_1(y)$.

Try this next

- 1 Agnew's problem.** Snow starts falling at a steady rate at an unknown time before noon. A snowplough whose speed is inversely proportional to the depth starts at noon and goes 2 miles in the first hour and 1 mile in the second. Show that if it started snowing t_0 hours before noon, then $(1 + \frac{2}{t_0})^2 = (1 + \frac{1}{t_0})^3$, deduce that $\frac{1}{t_0}$ is the golden ratio, and find when it started snowing.
- 2 Verification.** Verify by differentiation that $t_2(y)$ satisfies the differential equation of part (ii).
- 3 The time gap.** Show that the time gap $t_2(y) - t_1(y)$ first increases and then decreases to zero, and find where it is greatest. Interpret this.
- 4 Three ploughs.** A third plough starts at time $2T$. Write down the equation for its motion. Does it catch the second plough before or after the second catches the first?

Further reading

- ▶ **Ralph Palmer Agnew**, *Differential Equations* (McGraw-Hill, 1942). The original snowplough problem.
- ▶ **George Simmons**, *Differential Equations with Applications and Historical Notes* (CRC Press), which discusses the snowplough problem among many others.
- ▶ **Richard Haberman**, *Mathematical Models* (SIAM). Traffic flow and characteristics, at an accessible level.

The full guided solution, and what the question leads to at university, is in Section 11.13:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/snowplough.html