11.12 The Ant on the Rubber Rope

WORKSHEET

The question

A uniform elastic string lies on a smooth horizontal table. One end of the string is attached to a fixed peg, and the other end is pulled at constant speed u. At time $t = 0$, the string is taut and its length is a. Obtain an expression for the speed, at time t, of the point on the string which is a distance x from the peg at time t.

An ant walks along the string starting at $t = 0$ at the peg. The ant walks at constant speed v along the string (so that its speed relative to the peg is the sum of v and the speed of the point on the string beneath the ant). At time t, the ant is a distance x from the peg. Write down a first order differential equation for x, and verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}.$$

Show that the time T taken for the ant to reach the end of the string is given by

$$uT = a(e^k - 1),$$

where $k = u/v$.

On reaching the end of the string, the ant turns round and walks back to the peg. Find in terms of T and k the time taken for the journey back.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself The string is stretched uniformly: every piece is stretched by the same factor. At time t its length is $a + ut$. If the point at distance x from the peg is a fraction $\frac{x}{a+ut}$ of the way along, how fast does it move?

STAGE 1 The ant's equation

Ask yourself Differentiate $\frac{x}{a+ut}$ by the quotient rule and use the equation for $\dot{x}$.

STAGE 2 Reaching the end

Ask yourself Integrate from 0 to t. When does the fraction reach 1?

STAGE 3 The way back

Ask yourself Walking back, the ant's speed relative to the string is $-v$. What is the rate of change of its fraction now? It starts at fraction 1 at time T.

Try this next

1. **Halfway.** When is the ant halfway along the rope? Is that before or after time $\frac{T}{2}$?
2. **Exponential stretching.** Suppose instead the rope's length is $ae^{\lambda t}$. Show that the ant reaches the end only if $v > \lambda a$. (This is the cosmic event horizon.)
3. **An accelerating rope.** If the length is $a + \frac{1}{2}bt^2$, does the ant always arrive?
4. **The discrete puzzle.** In the discrete version, how many seconds does the ant need to cover 1% of the rope? Use $H_N \approx \ln N + 0.577$.

Further reading

- **Martin Gardner, *Time Travel and Other Mathematical Bewilderments*** (W. H. Freeman). Includes the ant on the rubber rope.
- **Julian Havil, *Gamma: Exploring Euler's Constant*** (Princeton University Press). The harmonic series, and the ant.
- **Barbara Ryden, *Introduction to Cosmology*** (Cambridge University Press). Expanding

space, horizons and comoving coordinates, at undergraduate level.

The full guided solution, and what the question leads to at university, is in Section 11.12:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/rubber-rope.html