

11.1 Resonance

WORKSHEET

The question

A particle is attached to one end B of a light elastic string of unstretched length a . Initially the other end A is at rest and the particle hangs at rest at a distance $a + c$ vertically below A . At time $t = 0$, the end A is forced to oscillate vertically, its downwards displacement at time t being $b \sin pt$. Let $x(t)$ be the downwards displacement of the particle at time t from its initial equilibrium position. Show that, while the string remains taut, $x(t)$ satisfies

$$\frac{d^2x}{dt^2} = -n^2(x - b \sin pt),$$

where $n^2 = g/c$, and that if $0 < p < n$, $x(t)$ is given by

$$x(t) = \frac{bn}{n^2 - p^2}(n \sin pt - p \sin nt).$$

Write down a necessary and sufficient condition that the string remains taut throughout the subsequent motion, and show that it is satisfied if $pb < (n - p)c$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** The particle hangs in equilibrium with extension c . If the string has modulus λ , what is the relation between λ , a , c , m and g ? What is the extension when the top has moved down by $b \sin pt$ and the particle by x ?

STAGE 2 Solving

- Ask yourself** The equation is $\ddot{x} + n^2x = n^2b \sin pt$. Try a particular solution of the form $x = C \sin pt$. Then add the general solution of $\ddot{x} + n^2x = 0$. Which initial conditions does the particle satisfy?

STAGE 3 Staying taut

- Ask yourself** The string is taut exactly when its extension is positive. Write the extension $c + x - b \sin pt$ in terms of t .

- Ask yourself** How negative can $p \sin pt - n \sin nt$ possibly be?

- Pause and try** What happens to this condition as p approaches n ? How small must the shaking amplitude b be when $p = 0.9n$? When $p = 0.99n$?

Try this next

- A falling lift.** (From STEP II, 1996.) A particle hangs from the ceiling of a lift by an elastic string, extended by a . The lift starts to descend with constant acceleration $f < \frac{g}{2}$. Show that the extension y of the string satisfies $\ddot{y} + \frac{g}{a}y = g - f$, that the string never becomes slack, and that the oscillation has amplitude $\frac{af}{g}$. After time T the lift stops accelerating. Show that the amplitude becomes $\frac{2af}{g} \left| \sin \frac{1}{2}\omega T \right|$, where $\omega^2 = \frac{g}{a}$.
- At resonance.** Solve $\ddot{x} + n^2x = n^2b \sin nt$ with $x(0) = \dot{x}(0) = 0$ directly, by trying a particular solution $x = Ct \cos nt$. Check that it agrees with the limit above.
- Beats.** Show that $n \sin pt - p \sin nt$ can be written as a sum of two terms, one of which is approximately $2n \cos \frac{(n+p)t}{2} \sin \frac{(p-n)t}{2}$ when p is close to n . Hence explain the beat period $\frac{2\pi}{n-p}$.

- 4 **Damped resonance.** For $\ddot{x} + 2k\dot{x} + n^2x = F \cos pt$ with small $k > 0$, find the steady-state amplitude, and show that it is greatest when $p^2 = n^2 - 2k^2$.

Further reading

- **A. P. French, *Vibrations and Waves*** (MIT Introductory Physics Series). Chapter 4 on forced oscillations and resonance is the classic account.
- **Steven Strogatz, *Sync*** (Penguin). How things fall into step, from fireflies to the Millennium Bridge.
- **Henry Petroski, *To Engineer Is Human*** (Vintage). Engineering failures, including the bridges in the history box.

The full guided solution, and what the question leads to at university, is in Section 11.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch11/resonance.html