10.8 Three Stars in a Triangle

WORKSHEET

The question

The force of attraction between two stars of masses m_1 and m_2 a distance r apart is $\gamma m_1 m_2 / r^2$. The Starmakers of Kryton place three stars of equal mass m at the corners of an equilateral triangle of side a. Show that it is possible for each star to revolve round the centre of mass of the system with angular velocity $(3\gamma m/a^3)^{1/2}$.

Find a corresponding result if the Starmakers place a fourth star, of mass λm, at the centre of mass of the system.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself For a star to move in a circle of radius R with angular velocity ω, what must the total force on it be?

STAGE 1 The geometry

Ask yourself How far is each corner of an equilateral triangle of side a from its centre?

STAGE 2 The force on one star

Ask yourself Each of the other two stars pulls with force $\frac{\gamma m^2}{a^2}$. What is the angle between the two pulls? What is their resultant, and in which direction does it point?

STAGE 4 A fourth star at the centre

Ask yourself A star of mass λm at the centre is at distance $\frac{a}{\sqrt{3}}$ from each corner. What extra force does it exert on each corner star? Does the centre of mass move? Does the central star feel any net force?

Try this next

1 **Four stars in a square.** Four stars of mass m are placed at the corners of a square of side a. Show that they can rotate rigidly about the centre with $\omega^2 = \frac{(4+\sqrt{2})\gamma m}{2a^3}$.

2 **Two equal stars and a planet.** Two stars of mass M orbit their centre of mass, a distance a apart. A planet of negligible mass sits at the centre. Is it in equilibrium? Is it stable to small displacements along the line joining the stars? Perpendicular to it?

3 **Unequal masses.** Three stars of masses m_1, m_2, m_3 are at the corners of an equilateral triangle of side a, rotating about their centre of mass G. Show that the gravitational force on m_1 is $\frac{\gamma m_1(m_1+m_2+m_3)}{a^3}\,\overrightarrow{P_1G}$, where P_1 is its position. (*Hint:* write the forces as vectors and use the definition of the centre of mass.) Deduce Lagrange's result.

4 **n stars.** n equal stars sit at the corners of a regular n-gon inscribed in a circle of radius R. Find ω^2 for rigid rotation as a sum over k of $\frac{1}{\sin(k\pi/n)}$.

Further reading

- **June Barrow-Green, *Poincaré and the Three Body Problem*** (American Mathematical Society). The story of the prize and Poincaré's discovery of chaos.
- **Richard Montgomery, "A new solution to the three-body problem"**, *Notices of the American Mathematical Society* 48 (2001). The figure eight, explained by one of its discoverers; free online.
- **Florin Diacu & Philip Holmes, *Celestial Encounters*** (Princeton University Press). A popular history of the three-body problem and chaos.

The full guided solution, and what the question leads to at university, is in Section 10.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch10/three-stars.html