

10.7 Gravity Inside a Shell

WORKSHEET

The question

The gravitational force between two point particles of masses m and m' is mutually attractive and has magnitude

$$\frac{Gmm'}{r^2},$$

where G is a constant and r is the distance between them.

A particle of unit mass lies on the axis of a thin uniform circular ring of radius r and mass m , at a distance x from its centre. Explain why the net force on the particle is directed towards the centre of the ring and show that its magnitude is

$$\frac{Gmx}{(x^2 + r^2)^{3/2}}.$$

The particle now lies inside a thin hollow spherical shell of uniform density, mass M and radius a , at a distance b from its centre. Show that the particle experiences no gravitational force due to the shell.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** Consider two small, diametrically opposite pieces of the ring.
 ? Draw the forces they exert on the particle. Which components cancel, and which add?

STAGE 2 Slicing the shell into rings

- Ask yourself** Put the centre of the shell at O and the particle on the axis at distance b from O . Slice the shell into thin rings perpendicular to the axis, labelled by the angle ϕ that a point of the ring makes with the axis at O . What are the radius, the position and the mass of the ring?

STAGE 3 The right variable

- Ask yourself** The denominator involves $s^2 = a^2 - 2ab \cos \phi + b^2$. What does s represent geometrically? What is ds in terms of $d\phi$?

STAGE 4 It vanishes

- Pause and try** What changes if the particle is *outside* the shell, with $b > a$? The limits of s become $b - a$ and $b + a$. Redo the last line and show that the force is $\frac{GM}{b^2}$ towards the centre, as if all the mass were concentrated there.

Try this next

- 1 Outside the shell.** Complete the Pause box: show that outside the shell the force is $\frac{GM}{b^2}$. Deduce that a solid uniform sphere of mass M pulls an outside particle as if its mass were at the centre.
- 2 The collapse time.** (From STEP III, 1987.) Using $v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$ and the substitution $r = R \cos^2 \alpha$, show that the time to reach radius r is $\sqrt{\frac{R^3}{2GM}}\left(\alpha + \frac{1}{2} \sin 2\alpha\right)$.
- 3 A tunnel through the Earth.** Inside a uniform solid sphere, show that the force at distance b from the centre is $\frac{GMb}{a^3}$, proportional to b . Deduce that a ball dropped down a tunnel through the centre of the Earth oscillates with period $2\pi\sqrt{a^3/(GM)}$, about 84 minutes, the same as the period of a satellite skimming the surface.
- 4 Another force law.** If gravity followed an inverse-cube law, would the force inside a

shell still be zero? (Repeat Stage 3 with $\frac{1}{s^3}$ in place of $\frac{1}{s^2}$.)

Further reading

- ▶ **Daniel Kleppner & Robert Kolenkow, *An Introduction to Mechanics*** (Cambridge University Press), the chapters on gravitation, including the shell theorem.
- ▶ **Subrahmanyan Chandrasekhar, *Newton's Principia for the Common Reader*** (Oxford University Press). Newton's proofs, including Propositions 70 and 71, explained by a Nobel laureate.
- ▶ **Andrew Pontzen, *The Universe in a Box*** (Riverhead). How gravity, collapse and simulation explain the structure of the universe.

The full guided solution, and what the question leads to at university, is in Section 10.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch10/shell-theorem.html