

# 10.6 Two Bodies Become One

## WORKSHEET

### The question

Particles  $P$ , of mass 2, and  $Q$ , of mass 1, move along a line. Their distances from a fixed point are  $x_1$  and  $x_2$ , respectively where  $x_2 > x_1$ . Each particle is subject to a repulsive force from the other of magnitude  $\frac{2}{z^3}$ , where  $z = x_2 - x_1$ .

Initially,  $x_1 = 0$ ,  $x_2 = 1$ ,  $Q$  is at rest and  $P$  moves towards  $Q$  with speed 1. Show that  $z$  obeys the equation  $\frac{d^2z}{dt^2} = \frac{3}{z^3}$ .

By first writing  $\frac{d^2z}{dt^2} = v \frac{dv}{dz}$ , where  $v = \frac{dz}{dt}$ , show that  $z = \sqrt{4t^2 - 2t + 1}$ .

By considering the equation satisfied by  $2x_1 + x_2$ , find  $x_1$  and  $x_2$  in terms of  $t$ .

## Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



**Ask yourself** Write down Newton's second law for each particle. In which direction does each force act?

### STAGE 1 The separation



**Ask yourself** Which combination of the two equations involves only  $z$ ?

### STAGE 2 Energy for the separation



**Ask yourself** With  $\ddot{z} = v \frac{dv}{dz}$ , separate the variables and integrate. What are the initial values of  $z$  and  $v = \dot{z}$ ?

### STAGE 3 A surprisingly simple solution



**Ask yourself** The answer to show is  $z^2 = 4t^2 - 2t + 1$ . So look at  $w = z^2$  instead of  $z$ . What is  $\dot{w}$  in terms of  $w$ ?

### STAGE 4 The centre of mass



**Ask yourself** What equation does  $2x_1 + x_2$  satisfy? Why is it the natural combination to look at?

## Try this next

- Two masses on a string.** (From STEP III, 2023.) Particles  $A$  (mass  $m$ ) and  $B$  (mass  $M$ ) are joined by a light string of length  $r$  on a smooth table. Initially  $A$  is at the origin and  $B$  at  $(r, 0)$ ;  $B$  is at rest and  $A$  is given velocity  $u$  in the  $y$  direction. Writing  $\theta$  for the angle of  $\overrightarrow{AB}$  clockwise from the  $x$ -axis, show that  $\theta = ut/r$ , and find  $my + MY$  in terms of  $t$ . Show that  $y = \frac{1}{m+M}(mut + Mr \sin \frac{ut}{r})$ , and that if  $M > m$  the  $y$ -velocity of  $A$  is sometimes negative.
- Attraction.** Repeat the question with an attractive force  $\frac{2}{z^3}$ . Show that  $z^2$  is still a quadratic in  $t$ , and find when the particles collide.
- The hidden plane.** For the motion in the question, find the straight line in the plane whose distance from the origin is  $z(t)$ , and the angle  $\theta(t)$  swept out. Check that  $z^2\dot{\theta}$  is constant and that  $(z^2\dot{\theta})^2 = 3$ .

- 4 Kepler's third law.** Two stars of masses  $m_1$  and  $m_2$  move in circles about their centre of mass, a distance  $a$  apart, under gravity. Show that the period  $T$  satisfies
- $$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}.$$

### Further reading

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- **John Taylor, *Classical Mechanics*** (University Science Books), Chapter 8. The two-body problem, reduced mass and orbits.
- **Daniel Kleppner & Robert Kolenkow, *An Introduction to Mechanics*** (Cambridge University Press), Chapter 10, on central forces.
- **Francesco Calogero, *Classical Many-Body Problems Amenable to Exact Treatments*** (Springer). For the ambitious: the world of exactly solvable systems.

The full guided solution, and what the question leads to at university, is in Section 10.6:  
[https://keithmcnulty.github.io/step\\_book\\_mcnulty/ch10/two-body.html](https://keithmcnulty.github.io/step_book_mcnulty/ch10/two-body.html)