

10.5 Infinitely Many Bounces in Finite Time

WORKSHEET

The question

A particle moves on a smooth triangular horizontal surface AOB with angle $AOB = 30^\circ$. The surface is bounded by two vertical walls OA and OB and the coefficient of restitution between the particle and the walls is e , where $e < 1$. The particle, which is initially at point P on the surface and moving with velocity u_1 , strikes the wall OA at M_1 , with angle $PM_1A = \theta$, and rebounds, with velocity v_1 , to strike the wall OB at N_1 , with angle $M_1N_1B = \theta$. Find e and $\frac{v_1}{u_1}$ in terms of θ .

The motion continues, with the particle striking side OA at $M_2, M_3, \dots$ and striking side OB at $N_2, N_3, \dots$. Show that, if $\theta < 60^\circ$, the particle reaches O in a finite time.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** When a particle hits a smooth wall with coefficient of restitution e , what happens to the component of velocity along the wall? And the component perpendicular to it?

STAGE 1 The angle of departure

- Ask yourself** The particle arrives at M_1 at angle θ to the wall OA (measured on the side away from O). At what angle φ to OA does it leave? Use the triangle OM_1N_1 to find φ in terms of θ .

STAGE 2 The pattern repeats

- Ask yourself** At N_1 , the particle arrives at angle θ to the wall OB , measured on the side away from O . Compare this with its arrival at M_1 . What happens next?

STAGE 3 Two ratios

- Ask yourself** By what factor ρ does the distance from O shrink from one bounce to the next? By what factor k does the speed shrink? So by what factor does the time for each leg shrink?

STAGE 4 A geometric series

- Ask yourself** When is $\sin(2\theta - 60^\circ) < \sin 2\theta$? Use the identity $\sin X - \sin Y = 2 \cos \frac{1}{2}(X + Y) \sin \frac{1}{2}(X - Y)$.

Try this next

- The total distance.** Show that the total distance travelled from M_1 to O is finite for every θ between 30° and 90° , and find it in terms of OM_1 and θ .
- Other wedges.** Repeat the question for a wedge of angle α instead of 30° . Show that $\varphi = \theta - \alpha$, and find the condition on θ for the particle to reach O in a finite time.
- Elastic walls.** If $e = 1$, can the particle ever reach O ? (*Hint:* reflect the wedge repeatedly in its walls, so that the path becomes a straight line. How many copies

of a 30° wedge fit round O ?)

- 4 The bouncing ball.** A ball is dropped from height 1 m with $e = 0.8$. How long does it bounce for, and how far does it travel in total? (Take $g = 9.8 \text{ m s}^{-2}$.)

Further reading

- **David Morin, *Introduction to Classical Mechanics*** (Cambridge University Press), Chapter 5. Includes several problems on bouncing balls and infinite sequences of collisions.
- **Sean McNamara & William Young, “Inelastic collapse and clumping in a one-dimensional granular medium”, *Physics of Fluids A* 4 (1992).**
- **Serge Tabachnikov, *Geometry and Billiards*** (American Mathematical Society; free online). The mathematics of billiards, including the unfolding trick in the exercises.

The full guided solution, and what the question leads to at university, is in Section 10.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch10/zeno-wedge.html