

10.4 A Molecule and a Piston

WORKSHEET

The question

A bicycle pump consists of a cylinder and a piston. The piston is pushed in with steady speed u . A particle of air moves to and fro between the piston and the end of the cylinder, colliding perfectly elastically with the piston and the end of the cylinder, and always moving parallel with the axis of the cylinder. Initially, the particle is moving towards the piston at speed v . Show that the speed, v_n , of the particle just after the n th collision with the piston is given by $v_n = v + 2nu$.

Let d_n be the distance between the piston and the end of the cylinder at the n th collision, and let t_n be the time between the n th and $(n + 1)$ th collisions. Express $d_n - d_{n+1}$ in terms of u and t_n , and show that

$$d_{n+1} = \frac{v + (2n - 1)u}{v + (2n + 1)u} d_n.$$

Express d_n in terms of d_1 , u , v and n .

In the case $v = u$, show that $ut_n = \frac{d_1}{n(n + 1)}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** A ball moving at speed w hits a wall that is moving towards it at speed u , and bounces elastically. What is its speed afterwards? (Think in the wall's frame.)

STAGE 2 A recurrence for the length

- Ask yourself** Between the n th and $(n + 1)$ th collisions, the particle goes from the piston to the end and back. How far does it travel? How far does the piston move?

STAGE 3 Telescoping

- Ask yourself** Write d_n as d_1 times a product of $n - 1$ such fractions. What cancels?

Try this next

- The invariant exactly.** Show that $v_n d_n - (v + u) d_1 = \frac{u(v+u)d_1}{v+(2n-1)u}$, and deduce that $v_n d_n \rightarrow (v + u) d_1$ as $n \rightarrow \infty$.
- Pulling the piston out.** Repeat the question with the piston withdrawn at speed $u < v$. Show that the particle slows down, and find $v_n d_n$. (Expanding gases cool.)
- Energy.** In the case $v = u$, the molecule's speed grows without limit as the piston approaches the end. Find the average force on the piston during the n th cycle (change of momentum divided by t_n) and show that it grows like n^3 . What does this say about the work needed to push the piston all the way in?
- A bouncing ball.** A ball bounces elastically on a floor under gravity, and the floor is raised slowly. Guess an adiabatic invariant for this motion. (*Hint:* the analogue of vd is speed times the time between bounces, or the height to the power $\frac{3}{2}$.)

Further reading

- **Richard Feynman, Robert Leighton & Matthew Sands, *The Feynman Lectures on Physics*, Vol. I** (free online), Chapter 39, "The kinetic theory of gases". The same piston argument, extended to three dimensions.
- **Stephen Blundell & Katherine Blundell, *Concepts in Thermal Physics*** (Oxford University

Press). Adiabatic processes and kinetic theory at first-year level.

- **Lev Landau & Evgeny Lifshitz, *Mechanics*** (Butterworth-Heinemann), §49. A concise treatment of adiabatic invariants, for later.

The full guided solution, and what the question leads to at university, is in Section 10.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch10/piston.html