

10.3 Rockets and Guns

WORKSHEET

The question

A railway truck, initially at rest, can move forwards without friction on a long straight horizontal track. On the truck, n guns are mounted parallel to the track and facing backwards, where $n > 1$. Each of the guns is loaded with a single projectile of mass m . The mass of the truck and guns (but not including the projectiles) is M .

When a gun is fired, the projectile leaves its muzzle horizontally with a speed $v - V$ relative to the ground, where V is the speed of the truck immediately before the gun is fired.

- (i) All n guns are fired simultaneously. Find the speed, u , with which the truck moves, and show that the kinetic energy, K , which is gained by the system (truck, guns and projectiles) is given by

$$K = \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right).$$

- (ii) Instead, the guns are fired one at a time. Let u_r be the speed of the truck when r guns have been fired, so that $u_0 = 0$. Show that, for $1 \leq r \leq n$,

$$u_r - u_{r-1} = \frac{mv}{M + (n - r)m} \quad (*)$$

and hence that $u_n < u$.

- (iii) Let K_r be the total kinetic energy of the system when r guns have been fired (one at a time), so that $K_0 = 0$. Using $(*)$, show that, for $1 \leq r \leq n$,

$$K_r - K_{r-1} = \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

and hence show that

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n.$$

Deduce that $K_n < K$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** Before a gun fires, the truck moves at V . Afterwards the projectile's velocity is $V - v$ (that is, speed $v - V$ backwards). What is the projectile's velocity *relative to the truck before firing*?

STAGE 2 One at a time

- Ask yourself** Just before the r th gun fires, what is the mass of the truck with its remaining projectiles, and what is its speed? Write down conservation of momentum for the r th firing.

- Pause and try** Why does firing one at a time do worse here? When the first gun fires, it has to accelerate the truck *and* the $n - 1$ unfired projectiles, so it produces less of a speed increase than it would on the bare truck.

STAGE 3 The energy bill

- Ask yourself** Write down the change in kinetic energy at the r th firing. How can (*) help you simplify the term involving $u_r^2 - u_{r-1}^2$?

Try this next

- The other convention.** Redo parts (i) and (ii) supposing each projectile leaves at speed v relative to the truck *after* firing. Show that now firing one at a time gives the larger final speed.
- A comet in a dust cloud.** (From STEP III, 2009.) A comet of mass M enters a stationary dust cloud at speed V ; after travelling a distance x its mass is $M(1 + bx)$. If no other force acts, show that $(1 + bx)v = V$ and find x in terms of t . If a constant force Mf acts in the direction of motion, use force = rate of change of momentum to show that $v = \frac{ft+V}{1+bx}$. Show that the comet can move at constant speed if $bV^2 = f$, and that otherwise its speed tends to $\sqrt{f/b}$.
- Two stages.** A rocket has exhaust speed v , empty mass M and fuel mass $9M$. Find its final speed. Now split it into two stages, each with empty mass $\frac{1}{2}M$ and fuel $4.5M$, the first stage being dropped when empty. Compare the final speeds.
- The inequality.** Prove that $\frac{x}{1+x} \leq \ln(1+x) \leq x$ for $x \geq 0$, by comparing $\frac{1}{1+s}$ with 1

and with $\frac{1}{1+x}$ on $[0, x]$.

Further reading

- ▶ **Daniel Kleppner & Robert Kolenkow, *An Introduction to Mechanics*** (Cambridge University Press), Chapter 4. Momentum, variable mass and the rocket equation.
- ▶ **George Sutton & Oscar Biblarz, *Rocket Propulsion Elements*** (Wiley). The standard engineering text; the opening chapters are readable after this section.
- ▶ **Hermann Oberth, *Ways to Spaceflight*** (NASA translation, free online). A pioneer's own account.

The full guided solution, and what the question leads to at university, is in Section 10.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch10/rockets.html