

10.1 Newton's Cradle

WORKSHEET

The question

The identical uniform smooth spherical marbles $A_1, A_2, \dots, A_n$, where $n \geq 3$, each of mass m , lie in that order in a smooth straight trough, with each marble touching the next. The marble A_{n+1} , which is similar to A_n but has mass λm , is placed in the trough so that it touches A_n . Another marble A_0 , identical to A_n , slides along the trough with speed u and hits A_1 . It is given that kinetic energy is conserved throughout.

- (i) Show that if $\lambda < 1$, there is a possible subsequent motion in which only A_n and A_{n+1} move (and A_0 is reduced to rest), but that if $\lambda > 1$, such a motion is not possible.
- (ii) If $\lambda > 1$, show that a subsequent motion in which only A_{n-1} , A_n and A_{n+1} move is not possible.
- (iii) If $\lambda > 1$, find a possible subsequent motion in which only two marbles move.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** A marble of mass m moving at speed u hits a stationary marble of mass M , and kinetic energy is conserved. What are the velocities afterwards? What if $M = m$?

STAGE 1 Only the last two move

- Ask yourself** Suppose only A_n (velocity p) and A_{n+1} (velocity q) move afterwards. Write down the conservation laws. Which marble's motion constrains the sign of p ?

STAGE 2 Three marbles

- Ask yourself** Now suppose only A_{n-1} , A_n and A_{n+1} move, with velocities a , b , c . What inequalities must a , b , c satisfy, given which marbles are at rest and which are behind which?

- Ask yourself** Square the momentum equation and compare it with the energy equation. What is left?

- Pause and try** Notice the structure of the argument. Squaring the momentum equation produces cross terms like $2ab$, and the energy equation has none. The difference measures how “spread out” the motion is. Can you see why the same argument rules out *any* outcome in which only $A_k, \dots, A_{n+1}$ move with non-negative velocities, when $\lambda > 1$?

STAGE 3 Two marbles that can move

- Ask yourself** If the marbles moving forwards must all be at the far end, and that fails, perhaps one marble moves *backwards*. Which marble is free to move backwards?

Try this next

- 1 The same problem again.** (From STEP I, 2006.) Particles $A_1, \dots, A_n$ lie at rest in a trough; A_{n-1} has mass m and A_n has mass λm with $\lambda > 1$. A particle A_0 of mass m hits A_1 with speed u . Show that it is impossible for exactly one particle to be

moving after all collisions, or for only A_{n-1} and A_n , or only A_{n-2} , A_{n-1} , A_n . Given that exactly two particles move, find their speeds.

- 2 **Following the collisions.** Model the impact in part (iii) as a sequence of separate two-marble collisions, starting with A_0 hitting A_1 . Follow the sequence through and show that it ends in the motion found in part (iii).
- 3 **Two balls in.** In a cradle of five equal balls, two balls are released together and hit the other three with speed u . Show that “two balls leave with speed u ” is consistent with both conservation laws, and find another outcome that is.
- 4 **Light end.** If $\lambda < 1$, follow the sequence of two-marble collisions. Do any marbles other than A_n and A_{n+1} end up moving?

Further reading

- **David Morin, *Introduction to Classical Mechanics*** (Cambridge University Press), Chapter 5. Conservation of momentum and energy, with many problems on collisions.
- **Frank Herrmann & Peter Schmälzle, “Simple explanation of a well-known collision experiment”**, *American Journal of Physics* 49 (1981). Why Newton’s cradle works.
- **Dwight Neuenschwander, *Emmy Noether’s Wonderful Theorem*** (Johns Hopkins University Press). An accessible account of the link between symmetry and conservation.

The full guided solution, and what the question leads to at university, is in Section 10.1:

https://keithmcnulty.github.io/step_book_mcnulty/ch10/newtons-cradle.html