

1.9 Tangents to a Cubic

WORKSHEET

The question

A curve has the equation

$$y^3 = x^3 + a^3 + b^3,$$

where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2y - a^2x = a^3 + b^3.$$

In the case $a = 1$ and $b = 2$, show that the x -coordinates of the points where the tangent meets the curve satisfy

$$7x^3 - 3x^2 - 27x - 17 = 0.$$

Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** The point $(-a, b)$ is on the curve. Why? And what does that have to do with sums of cubes?

STAGE 3 Solving the cubic

? **Ask yourself** You already know one root of this cubic without doing any algebra. Which one, and why should it appear *twice*?

✓ **Pause and try** So the tangent meets the curve again where $x = \frac{17}{7}$. Find y , and turn the point into a relation between cubes.

? **Ask yourself** We want 9×7^3 to be a sum of two cubes. Remember where the 9 came from.

Try this next

- 1 Rational points on a circle.** The line through $(-1, 0)$ with gradient t meets the circle $x^2 + y^2 = 1$ again. Show that the other point is $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$, and use this to produce infinitely many Pythagorean triples.
- 2 A chord that misses.** The points $(1, 2)$ and $(2, 1)$ lie on $x^3 + y^3 = 9$. Show that the line through them meets the curve in no other point. Where has the “third point” gone? (Look up the *point at infinity*.)
- 3 Repeat the trick.** Find the tangent to $y^3 = x^3 + 9$ at $\left(\frac{17}{7}, \frac{20}{7}\right)$ and the third point where it meets the curve. (A computer algebra system helps; notice how quickly the numbers grow.)
- 4 Taxicab numbers.** $1729 = 1^3 + 12^3 = 9^3 + 10^3$. Find the third point where the line through $(1, 12)$ and $(9, 10)$ meets the curve $x^3 + y^3 = 1729$, and deduce a new identity of the form $p^3 = q^3 + r^3 + s^3$.

Further reading

- **Joseph Silverman & John Tate, *Rational Points on Elliptic Curves*** (Springer). Written for undergraduates; Chapter 1 needs nothing beyond A-level.

- **Avner Ash & Robert Gross, *Elliptic Tales*** (Princeton University Press). A popular account leading up to the Birch and Swinnerton-Dyer conjecture.
- **Simon Singh, *Fermat's Last Theorem***. How elliptic curves came to solve a 350-year-old problem.

The full guided solution, and what the question leads to at university, is in Section 1.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/tangents-cubic.html