

1.8 Counting the Rationals

WORKSHEET

The question

A sequence u_k , for integer $k \geq 1$, is defined as follows.

$$u_1 = 1, \quad u_{2k} = u_k \text{ for } k \geq 1, \quad u_{2k+1} = u_k + u_{k+1} \text{ for } k \geq 1.$$

- (i) Show that, for every pair of consecutive terms of this sequence, except the first pair, the term with odd subscript is larger than the term with even subscript.
- (ii) Suppose that two consecutive terms in this sequence have a common factor greater than one. Show that there are then two consecutive terms earlier in the sequence which have the same common factor. Deduce that any two consecutive terms in this sequence are co-prime (do not have a common factor greater than one).
- (iii) Prove that it is not possible for two positive integers to appear consecutively in the same order in two different places in the sequence.
- (iv) Suppose that a and b are two co-prime positive integers which do not occur consecutively in the sequence with b following a . If $a > b$, show that $a - b$ and b are two co-prime positive integers which do not occur consecutively in the sequence with b following $a - b$, and whose sum is smaller than $a + b$. Find a similar result for $a < b$.
- (v) For each integer $n \geq 1$, define the function f from the positive integers to the positive rational numbers by $f(n) = \frac{u_n}{u_{n+1}}$. Show that the range of f is all the positive rational numbers, and that f has an inverse.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Before proving anything, compute the first fifteen or so terms. What do you notice about the ratios u_n/u_{n+1} ?

STAGE 1 Odd beats even



Ask yourself Write each of these pairs in terms of u_k and u_{k+1} . Which term is bigger?

STAGE 2 Consecutive terms are coprime



Ask yourself Suppose $d > 1$ divides both u_{2k} and u_{2k+1} , that is, both u_k and $u_k + u_{k+1}$. What else does d divide?

STAGE 3 No pair appears twice



Pause and try Stage 1 tells you something useful: if you see a consecutive pair (a, b) , you can tell whether a sits at an odd or an even position. How? And can you then recover the pair that came before it, in the sense of Stage 2?

STAGE 4 Every coprime pair appears



Ask yourself Suppose some coprime pair (a, b) never appears consecutively. Which such pair should you look at?

Try this next

- Find the fraction.** For which n is $f(n) = \frac{3}{5}$? And $f(n) = \frac{5}{3}$? Use the parent rule to work backwards to $\frac{1}{1}$.
- Newman's formula.** Check that Newman's formula produces the first ten fractions of the list, starting from $x_1 = 1$.
- Cantor's diagonal argument.** Suppose you had a list of all real numbers between 0 and 1, written as decimals. Construct a decimal that differs from the n th number in the list in its n th digit. Why does this show the list is incomplete? (What care is needed with decimals like $0.0999 \dots = 0.1$?)

- 4 **Countable unions.** **HARDER** Show that the set of all pairs (m, n) of positive integers is countable by giving an explicit bijection with the positive integers. (Try $2^{m-1}(2n-1)$.)

Further reading

- ▶ **Neil Calkin & Herbert Wilf**, “**Recounting the rationals**” (*American Mathematical Monthly*, 2000). Four pages, entirely readable.
- ▶ **Aigner & Ziegler**, ***Proofs from THE BOOK***, Chapter 17, “Sets, functions, and the continuum hypothesis”, which includes the Calkin–Wilf tree.
- ▶ **Ian Stewart & David Tall**, ***The Foundations of Mathematics*** (Oxford University Press). Countability, Cantor and the real numbers, as in a first-year course.

The full guided solution, and what the question leads to at university, is in Section 1.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/counting-rationals.html