

1.6 Sums of Two Squares

WORKSHEET

The question

- (i) The complex numbers z and w have real and imaginary parts given by $z = a + ib$ and $w = c + id$. Prove that $|zw| = |z||w|$.
- (ii) By considering the complex numbers $2 + i$ and $10 + 11i$, find positive integers h and k such that $h^2 + k^2 = 5 \times 221$.
- (iii) Find positive integers m and n such that $m^2 + n^2 = 8045$.
- (iv) You are given that $102^2 + 201^2 = 50805$. Find positive integers p and q such that $p^2 + q^2 = 36 \times 50805$.
- (v) Find three distinct pairs of positive integers r and s such that $r^2 + s^2 = 25 \times 1002082$ and $r < s$.
- (vi) You are given that $109 \times 9193 = 1002037$. Find positive integers t and u such that $t^2 + u^2 = 9193$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** If $z = a + ib$ with a and b integers, what is $|z|^2$?

STAGE 1 Moduli multiply

? **Ask yourself** Write out zw in terms of a, b, c, d . What do you need to show about $|zw|^2$?

STAGE 2 Using it

? **Ask yourself** What happens if you multiply $10 + 11i$ by $2 - i$ instead? It has the same modulus as $2 + i$.

STAGE 3 Factorise first

? **Ask yourself** 8045 ends in 5. What does that suggest?

STAGE 4 Three ways

? **Ask yourself** 25 can be written as a sum of two squares in more than one way. How many Gaussian integers have modulus 5, up to signs and swapping?

STAGE 5 Dividing

✓ **Pause and try** Part (vi) runs the machine backwards. We know $109 = 10^2 + 3^2$ and $1002037 = 1001^2 + 6^2$. How might you find 9193 as a sum of two squares, without searching?

? **Ask yourself** What happens if you divide by $10 - 3i$ instead?

Try this next

- 1 An obstruction.** Show that no number of the form $4k + 3$ is a sum of two squares. (What are the squares modulo 4?)
- 2 Two ways.** Write 65 and 85 each as a sum of two squares in two different ways, and

explain your answers using Gaussian integers.

- 3 Four ways.** $1105 = 5 \times 13 \times 17$. Find four different ways to write 1105 as a sum of two positive squares. Why four?
- 4 Gaussian primes.** **HARDER** Show that if $3 = (a + bi)(c + di)$ with a, b, c, d integers, then one of the factors has modulus 1. (Take moduli squared.) This shows that 3 is a Gaussian prime.

Further reading

- ▶ **John Stillwell**, *Elements of Number Theory* (Springer). Builds up to Gaussian integers and beyond, with history.
- ▶ **Aigner & Ziegler**, *Proofs from THE BOOK*, Chapter 4, “Representing numbers as sums of two squares”.
- ▶ **Joseph Silverman**, *A Friendly Introduction to Number Theory*, chapters on sums of squares and Gaussian integers.

The full guided solution, and what the question leads to at university, is in Section 1.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/two-squares.html