

1.5 Infinite Descent

WORKSHEET

The question

- (i) Suppose that there are three non-zero integers a , b and c for which $a^3 + 2b^3 + 4c^3 = 0$.

Explain why there must exist an integer p , with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers p , q and r , with $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$, such that $p^3 + 2q^3 + 4r^3 = 0$.

Deduce that no such integers a , b and c can exist.

- (ii) Prove that there are no non-zero integers a , b and c for which $9a^3 + 10b^3 + 6c^3 = 0$.

- (iii) By considering the expression $(3n \pm 1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of 3. Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers a , b and c for which $a^2 + b^2 = 3c^2$.

- (iv) Prove that there are no non-zero integers a , b and c for which $a^2 + b^2 + c^2 = 4abc$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself In part (i), look at $a^3 + 2b^3 + 4c^3 = 0$. Two of the three terms are obviously even. What does that tell you about the third?

STAGE 1 The first descent



Ask yourself The new equation is $4p^3 + b^3 + 2c^3 = 0$. It has the same coefficients, just rotated. Can you do the same trick again?



Pause and try Turn this into a proof that no non-zero solution exists. What, exactly, should you assume is the “smallest”?

STAGE 2 A different prime



Ask yourself In $9a^3 + 10b^3 + 6c^3 = 0$, parity doesn't obviously help. Look at the coefficients 9, 10, 6. Which prime divides two of them?

STAGE 3 Squares modulo 3



Ask yourself Every integer is of the form $3n$, $3n + 1$ or $3n - 1$. What are their squares, modulo 3?



Ask yourself If $a^2 + b^2 = 3c^2$ and $a = 3p$, $b = 3q$, what can you say about c ?

STAGE 4 A twist: the equation changes



Ask yourself In $a^2 + b^2 + c^2 = 4abc$ the right-hand side is a multiple of 4. What are the squares modulo 4?



Ask yourself This is *not* the same equation: the 4 has become an 8. Does the descent still work?

Try this next

- 1 **A STEP I warm-up.** (*STEP I 2010 Question 8.*) Show that the only integer solution of $a^3 + 3b^3 = 9c^3$ is $a = b = c = 0$. Then, by considering final digits, show the same for $p^4 + 2q^4 = 5r^4$.
- 2 **A modulus with no descent needed.** Show that $x^2 + y^2 = 4z + 3$ has no integer solutions at all, just by working modulo 4.
- 3 **Rational points.** Use part (iii) to show that there is no point with *rational* coordinates on the circle $x^2 + y^2 = 3$. (Contrast this with $x^2 + y^2 = 2$, which contains $(1, 1)$.)
- 4 **Markov triples.** Show that if (x, y, z) satisfies $x^2 + y^2 + z^2 = 3xyz$, then so does $(x, y, 3xy - z)$. Starting from $(1, 1, 1)$, generate the triples listed above. (This “jumping” trick is the opposite of descent.)

Further reading

- **Harold Davenport**, *The Higher Arithmetic*, Chapters 2 and 5.
- **Joseph Silverman**, *A Friendly Introduction to Number Theory* (Pearson). Very readable; includes Fermat’s descent proof for $x^4 + y^4 = z^2$.
- **Simon Singh**, *Fermat’s Last Theorem* (Fourth Estate). The story, from Fermat’s margin to Wiles’s proof, for a general reader.

The full guided solution, and what the question leads to at university, is in Section 1.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/infinite-descent.html