1.4 e Is Irrational, and So Is e^2

WORKSHEET

This section uses two questions. The first, from 1997, proves that e is irrational. The second, from 2024, uses a cleverer version of the same idea to prove that e^2 is irrational, which turns out to be genuinely harder.

Warm-up: STEP III 1997, Question 7

For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots,$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots.$$

(i) Evaluate b_n.

(ii) Show that $0 < a_n < 1/n$.

(iii) Deduce that $a_n = n!\,e - [n!\,e]$ (where $[x]$ is the integer part of x).

(iv) Hence show that e is irrational.

Main question: STEP III 2024, Question 7

In this question, you need not consider issues of convergence.

For positive integer n let

$$f(n) = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

and

$$g(n) = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} - \cdots.$$

(i) Show, by considering a geometric series, that $0 < f(n) < \dfrac{1}{n}$.

(ii) Show, by comparing consecutive terms, that $0 < g(n) < \dfrac{1}{n+1}$.

(iii) Show, for positive integer n, that $(2n)!\,e - f(2n)$ and $\dfrac{(2n)!}{e} + g(2n)$ are both integers.

(iv) Show that if $qe = \dfrac{p}{e}$ for some positive integers p and q, then $qf(2n) + pg(2n)$ is an integer for all positive integers n.

(v) Hence show that the number e^2 is irrational.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Suppose $e = p/q$. Multiplying e by q gives an integer. What does multiplying by $q!$ do to each term of the series for e?

STAGE 1 The warm-up: e is irrational

Ask yourself Compare a_n and b_n term by term. Which is bigger?

Pause and try Now finish the proof. Suppose $e = p/q$. Which value of n should you choose?

STAGE 2 Why e^2 is harder

Ask yourself Try the same idea on e^2. Its series is $\sum 2^k/k!$. What goes wrong when you multiply by $n!$ and look at the tail?

STAGE 3 The two tails

Ask yourself For an alternating series $t_1 - t_2 + t_3 - \cdots$ with $t_1 > t_2 > t_3 > \cdots > 0$, why is the sum between 0 and t_1?

STAGE 4 Integers again

Ask yourself What is the sign of the first term of the tail, $k = 2n + 1$? How does the tail compare with $g(2n)$?

STAGE 5 The contradiction

Pause and try Now use the bounds from Stage 3. What happens to $qf(2n) + pg(2n)$ when n is very large?

Try this next

1 **Using $1/e$ alone.** Prove that e is irrational by supposing $1/e = p/q$ and using only the alternating series for $1/e$.

2 $\sin 1$ **is irrational.** Use $\sin 1 = 1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \cdots$ to prove that $\sin 1$ is irrational. (Multiply by a suitable odd factorial.)

3 **Why not π?** Explain why this method doesn't work directly for π. Then look up Ivan Niven's one-page proof (1947) that π is irrational, which uses an integral instead of a series. It is within reach of a strong A-level student.

4 **One of them is transcendental.** HARDER Show that $e + \pi$ and $e\pi$ are the roots of $x^2 - (e+\pi)x + e\pi = 0$. You may assume that a root of a polynomial whose coefficients are algebraic numbers is itself algebraic, and that π is not algebraic (Lindemann). Deduce that at least one of $e + \pi$ and $e\pi$ is transcendental.

Further reading

- **Aigner & Ziegler, *Proofs from THE BOOK*,** Chapter 8, "Some irrational numbers". Proves that e, e^2, e^4 and π^2 are irrational, using exactly these ideas.
- **Ivan Niven, *Irrational Numbers*** (Mathematical Association of America). A short classic covering everything from $\sqrt{2}$ to transcendence.
- **Julian Havil, *The Irrationals*** (Princeton University Press). A readable history of irrational numbers.

The full guided solution, and what the question leads to at university, is in Section 1.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/e-irrational.html