

1.3 The Smallest Counterexample

WORKSHEET

The question

- (i) In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.
1. Assume that $\sqrt{2}$ is rational.
 2. Define the set S to be the set of positive integers with the following property: n is in S if and only if $n\sqrt{2}$ is an integer.
 3. Show that the set S contains at least one positive integer.
 4. Define the integer k to be the smallest positive integer in S .
 5. Show that $(\sqrt{2} - 1)k$ is in S .
 6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.
- (ii) Prove that $2^{1/3}$ is rational if and only if $2^{2/3}$ is rational.
- Use an argument similar to that of part (i) to prove that $2^{1/3}$ and $2^{2/3}$ are irrational.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Step 4 says “define k to be the smallest positive integer in S ”. Why are we entitled to do that? Would it be allowed if S were a set of positive *real* numbers?

STAGE 1 Step 3: S is not empty

Ask yourself If $\sqrt{2} = a/b$ for positive integers a and b , can you write down an element of S ?

STAGE 2 Step 5: a smaller element

Pause and try To show that $(\sqrt{2} - 1)k$ belongs to S you must check three things. What are they?

STAGE 3 Step 6: the contradiction

Ask yourself How big is $\sqrt{2} - 1$?

STAGE 4 Cube roots: linking the two numbers

Ask yourself If c is rational, is c^2 ? And if c^2 is rational, can you get back to c using only multiplication and division?

STAGE 5 Adapting the argument

Ask yourself In part (i), the key was that $(\sqrt{2})^2 = 2$ is an integer, so multiplying an element of S by $\sqrt{2}$ twice takes you back to integers. For $c = 2^{1/3}$ you would need to multiply *three* times. How should the set S change?

Try this next

- Other square roots.** Adapt the argument to show that $\sqrt{3}$ is irrational. Which number plays the role of $\sqrt{2} - 1$?
- All non-squares.** Let N be a positive integer that is not a perfect square, and let m

be the integer with $m < \sqrt{N} < m + 1$. Prove that $\sqrt{N}$ is irrational by considering $(\sqrt{N} - m)k$.

3 Spot the flaw. Try to run the argument for $\sqrt{4}$. Exactly which step fails?

4 Tennenbaum's picture. Use Figure 1.3.1 (in the full section online) to show that if $a^2 = 2b^2$ then $(2b - a)^2 = 2(a - b)^2$. Check this identity algebraically too.

Further reading

- **Daniel Velleman, *How to Prove It*** (Cambridge University Press). A thorough introduction to proof, including induction and well-ordering.
- **R. P. Burn, *A Pathway into Number Theory*** (Cambridge University Press). A book made entirely of carefully sequenced questions; ideal preparation for a first-year course.
- **John Conway & Joseph Shipman, "Extreme proofs I: the irrationality of $\sqrt{2}$ "** (*Mathematical Intelligencer*, 2013). Many proofs of one fact, compared with wit.

The full guided solution, and what the question leads to at university, is in Section 1.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/well-ordering.html