

1.2 How Thinly Spread Are the Primes?

WORKSHEET

The question

- (i) By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m}$$

for any positive integer m .

- (ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s , if there are any such prime numbers; if there are no such prime numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m , $P_{m+1,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1,2m+1} < 2^{2m}.$$

- (iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \dots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.
- (iv) Prove that $P_{1,n} < 4^n$ for $n \geq 2$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Part (iv) asks you to prove a statement for *all* $n \geq 2$. Part (iii) assumes the statement for every k up to $2m$. What kind of induction is that?

STAGE 1 A binomial coefficient is not too big



Ask yourself What do you get if you put $x = 1$ in the binomial expansion of $(1 + x)^{2m+1}$?



Ask yourself Pascal's triangle is symmetric. Which other term in the sum equals $\binom{2m+1}{m}$?

STAGE 2 Which primes divide that binomial coefficient?



Ask yourself Let p be a prime with $m + 1 < p \leq 2m + 1$. Does p divide the numerator? Can it divide the denominator?



Pause and try We now know that *each* prime in the range divides $\binom{2m+1}{m}$. Why does it follow that their *product* $P_{m+1, 2m+1}$ divides it too? (Careful: if 4 and 6 both divide a number, 24 need not.)

STAGE 3 Splitting the product



Ask yourself How can $P_{1, 2m+1}$, the product of all primes up to $2m + 1$, be built from products we already know about?

STAGE 4 All n



Ask yourself Part (iii) handles odd numbers. What happens for an even number $n = 2m + 2$? Is it prime?

Try this next

- 1 Feel the numbers.** Compute $P_{1, n}$ for $n = 10, 20, 30$ and compare it with 4^n and with e^n . (A calculator is fine here.)

2 The middle binomial coefficient. Show that $\binom{2m}{m}$ is the largest of the $2m + 1$ terms in the expansion of $(1 + 1)^{2m}$, and deduce that $\binom{2m}{m} \geq \frac{4^m}{2m + 1}$.

3 Not too many primes. Let $\pi(n)$ be the number of primes up to n . Every prime between $\sqrt{n}$ and n is bigger than $\sqrt{n}$, and their product is at most $P_{1,n} < 4^n$. Deduce that

$$\pi(n) - \pi(\sqrt{n}) < \frac{2 \ln 4}{\ln n} n,$$

and hence that $\pi(n) < 2.8n / \ln n + \sqrt{n}$. Compare this with the Prime Number Theorem's $\pi(n) \approx n / \ln n$.

4 Check Bertrand. Verify Bertrand's postulate for $n = 1, 2, \dots, 50$ by finding a prime between n and $2n$ in each case. Can you do it efficiently using a short list of primes, each less than twice the one before?

Further reading

- **Aigner & Ziegler, *Proofs from THE BOOK***, Chapter 2, "Bertrand's postulate".
- **Marcus du Sautoy, *The Music of the Primes*** (Fourth Estate). A popular history of the Prime Number Theorem and the Riemann Hypothesis.
- **G. J. O. Jameson, *The Prime Number Theorem*** (Cambridge University Press). Rigorous but gentle, and written for undergraduates; the first chapters need only A-level calculus.

The full guided solution, and what the question leads to at university, is in Section 1.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch01/prime-distribution.html