

CHAPTER

8

Differential Equations and Dynamical Systems

Most differential equations cannot be solved. The art is to understand them anyway.

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Rules for Change

Nature laughs at the difficulties of integration.

— Attributed to Pierre-Simon Laplace

At school, a differential equation is something to be solved. You separate the variables, or find an integrating factor, or guess a solution of the form $e^{\lambda x}$, and at the end you have a formula. The equations in textbooks are chosen because this works. Unfortunately, almost every differential equation that comes from the real world (the motion of three planets, the weather, a population of predators and prey, a swinging pendulum with a large amplitude) has no solution that can be written down with the functions we know.

This chapter is about what to do instead. Rather than asking “what is the formula?”, it asks questions that can be answered without one. Is there a solution at all, and is there only one? Does it stay bounded? Does it settle down to a steady state, or to a repeating cycle, or does it wander for ever? What does it do in the long run? This change of viewpoint, from solving to understanding, was begun by Henri Poincaré in the 1880s, and it created the subject now called *dynamical systems*.

Why differential equations matter

Differential equations are the language of science. Newton’s second law is a differential equation; so are Maxwell’s equations of electromagnetism, the Schrödinger equation of quantum mechanics, and the equations of fluid flow used to forecast the weather and design aircraft. Epidemiologists model the spread of a disease with them, ecologists the populations of fish stocks, economists the growth of an economy, and engineers the response of every control system from a thermostat to an autopilot. Whenever a quantity changes at a rate that depends on its current state, there is a differential equation, and usually one that nobody can solve exactly.

Alongside these continuous models are discrete ones, where time moves in steps: the population this year determines the population next year. These *iterated maps* are simpler to write down, but the last two sections show that they can behave just as wildly. One of the great discoveries of the 1970s was that very simple rules can produce *chaos*: behaviour that is completely determined, yet in practice unpredictable.

What you will meet at university

A first-year course on differential equations covers the standard methods of solution, but it also introduces the questions of this chapter. You will meet existence and uniqueness theorems, which tell you when a solution exists and when it is determined by its starting value. You will learn to sketch *phase portraits*, pictures of all solutions at once,

and to classify equilibria as stable or unstable by linearising. Later courses go much further: *nonlinear dynamics* studies limit cycles, bifurcations and chaos; *mathematical biology* applies them to populations and epidemics; and the theory of partial differential equations, Fourier series and orthogonal functions (glimpsed in Section 8.6) underpins quantum mechanics and much of applied mathematics.

The questions that run through this chapter

Three big questions

1. **Does the present determine the future?** When does a differential equation, with a starting value, have exactly one solution?
2. **What can we know without solving?** How can we bound a solution, find where it settles, or describe its shape, when no formula exists?
3. **What happens in the long run?** Do solutions settle to a steady state, approach a cycle, or behave chaotically, and how can we tell?

The chapter begins with uniqueness and what happens when it fails (Section 8.1), then develops tools for understanding equations without solving them: energy-like quantities that cannot increase, equilibria and their stability, and coordinates matched to the symmetry of the problem. Two sections apply these ideas to physical problems: dogs chasing each other in a square, and the quantum harmonic oscillator. The chapter ends with discrete maps, where a formula reveals chaos, and where a single cycle of length three forces cycles of every length.

Ideas in this chapter

§	Topic	Idea
8.1	Uniqueness	Envelopes and singular solutions; gluing
8.2	Energy methods	A quantity that never increases
8.3	Population models	Equilibria, stability and a tipping point
8.4	Limit cycles	Polar coordinates and the Bernoulli substitution
8.5	Pursuit curves	Symmetry reduces four bodies to one
8.6	Orthogonal functions	The Wronskian and orthogonality
8.7	Chaos	Conjugacy: the logistic map is angle doubling
8.8	Periodic orbits	Covering intervals; Sharkovskii's theorem

How to read this chapter

Before solving anything, sketch. For a differential equation $y' = F(y)$, mark where F is positive, negative and zero, and draw arrows on the y -axis; for a map $x \mapsto f(x)$, draw its graph with the line $y = x$ and trace a few sequences by the cobweb method. These pictures take a minute, and they will usually tell you what the answer should look like before the algebra begins.

8.1 When Solutions Aren't Unique

The question

Find the two solutions of the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 4y$$

which pass through the point (a, b^2) , where $b \neq 0$.

Find two distinct points $(a_1, 1)$ and $(a_2, 1)$ such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1, 1)$ and $(a_2, 1)$.

Now sketch sufficient members of the family of solutions (for varying a and b) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

The hook

A differential equation is a rule that tells you how a quantity changes: Newton's laws tell a planet how to move, given where it is now. The classical picture of physics is that this is enough: know the state of the world now, and the laws determine its entire future. Pierre-Simon Laplace imagined an intellect that, knowing every position and velocity, could predict everything. That picture rests on a mathematical assumption: that a differential equation, with a starting condition, has *exactly one* solution.

This question shows that the assumption can fail. The innocent-looking equation $(y')^2 = 4y$ has solutions that run along the x -axis and then, at a moment of their own choosing, peel away. Through some points pass infinitely many solutions. Knowing the present does not determine the future. Finding out exactly when uniqueness holds is one of the first theorems of a university course on differential equations.

Before you start



Ask yourself Take the square root of both sides: $y' = \pm 2\sqrt{y}$. Try $y = (x - c)^2$. Does it satisfy the equation? Is there an even simpler solution?

If $y = (x - c)^2$, then $y' = 2(x - c)$ and $(y')^2 = 4(x - c)^2 = 4y$. So every parabola $y = (x - c)^2$ is a solution, for every x . And $y = 0$ is also a solution: $(0)^2 = 4 \times 0$. Keep that last one in mind; it is the key to the whole question.

Working it through

STAGE 1 Solutions through a point



Ask yourself Which parabolas $y = (x - c)^2$ pass through (a, b^2) ?

We need $(a - c)^2 = b^2$, so $a - c = \pm b$, that is, $c = a - b$ or $c = a + b$. Since $b \neq 0$ these are different, giving the two solutions

$$y = (x - a + b)^2 \quad \text{and} \quad y = (x - a - b)^2. \quad \blacksquare$$

(We should also check that we have not missed any: on a region where $y > 0$, $y' = \pm 2\sqrt{y}$ with a fixed sign, and $\frac{d}{dx}\sqrt{y} = \frac{y'}{2\sqrt{y}} = \pm 1$, so $\sqrt{y} = \pm(x - c)$, giving these parabolas.)

STAGE 2 Two points and a common solution

A solution through the origin must be $y = x^2$ (the parabola with $c = 0$). It passes through height 1 at $x = 1$ and $x = -1$. So take $(a_1, 1) = (1, 1)$ and $(a_2, 1) = (-1, 1)$.

Through $(1, 1)$ (so $a = 1$, $b = \pm 1$), the solutions are $y = x^2$ and $y = (x - 2)^2$. Through $(-1, 1)$ they are $y = (x + 2)^2$ and $y = x^2$. The solution through the origin is the same curve $y = x^2$ in both cases. $\blacksquare$

STAGE 3 The whole family, and a common tangent



Pause and try Sketch several of the parabolas $y = (x - c)^2$ for different c . Which line touches every one of them? Is that line itself a solution?

Every parabola $y = (x - c)^2$ touches the x -axis at its vertex $(c, 0)$, so the x -axis is a common tangent to the whole family: their *envelope* (Figure 8.1.1). And the x -axis, $y = 0$, is itself a solution of the differential equation. It is not one of the parabolas; it is a separate, *singular* solution.

STAGE 4 Why it matters



Ask yourself Start at the origin. Which solutions pass through it? Can you build a solution that follows the x -axis for a while and then leaves it?

Through the origin pass $y = x^2$ and $y = 0$. But much more is true. Because each parabola meets the x -axis tangentially (both have gradient 0 there), you can glue pieces together smoothly: for any $c_1 \leq 0 \leq c_2$,

$$y = \begin{cases} (x - c_1)^2 & x < c_1, \\ 0 & c_1 \leq x \leq c_2, \\ (x - c_2)^2 & x > c_2 \end{cases}$$

is a differentiable function satisfying $(y')^2 = 4y$ everywhere, and it passes through the origin. There are infinitely many solutions through every point on the x -axis: a solution can sit at $y = 0$ for as long as it likes, and then leave. The common tangent is exactly where uniqueness breaks down. ■

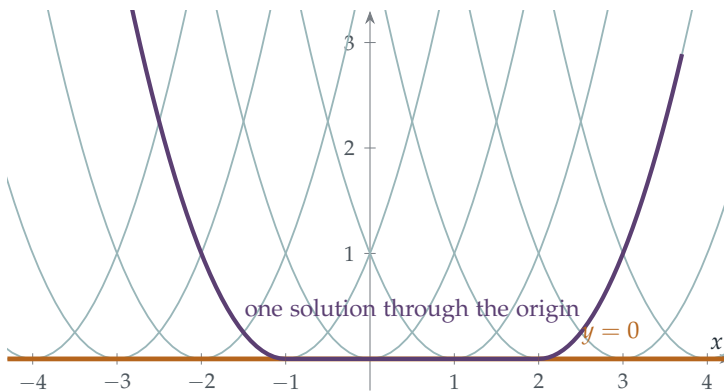


Figure 8.1.1 The solutions $y = (x - c)^2$ of $(y')^2 = 4y$ (teal) all touch the x -axis, which is their envelope and is itself a solution (amber). Gluing pieces gives further solutions, such as the plum one, which follows a parabola down to the axis, runs along it, and leaves again at a time of its choosing. Through any point of the x -axis pass infinitely many solutions.

Looking back

- **Look for simple solutions first.** Trying $y = (x - c)^2$ and $y = 0$ found the whole picture.
- **Envelopes give singular solutions.** The common tangent to a family of solutions can itself be a solution that is not in the family.
- **Gluing.** Where two solutions touch tangentially, they can be joined into a new differentiable solution. This is how non-uniqueness spreads.

What's really going on

When are solutions unique?

The *Picard–Lindelöf theorem* (proved by Émile Picard and Ernst Lindelöf around 1890–1894) says that the equation $y' = F(x, y)$ with $y(x_0) = y_0$ has exactly one solution near x_0 , provided F is *Lipschitz* in y : there is a constant L with $|F(x, y_1) - F(x, y_2)| \leq L|y_1 - y_2|$. Roughly, F must not change infinitely fast as y changes. Our equation is $y' = \pm 2\sqrt{y}$, and $\sqrt{y}$ has infinite slope at $y = 0$: it is not Lipschitz there, and that is exactly where uniqueness fails. (Giuseppe Peano showed in 1886 that continuity of F alone guarantees that a solution *exists*, but not that it is unique.) The proof of Picard–Lindelöf uses an iteration that converges by the contraction mapping principle of Section 2.2.

The leaking bucket

Here is a physical example. A bucket with a hole in the bottom empties according to Torricelli's law: the height h of water satisfies $h' = -k\sqrt{h}$. Run time forwards and everything is fine: the bucket empties and stays empty. But run time backwards: if you find an empty bucket, the equation cannot tell you when it emptied. The solution $h = 0$ for all time and the solutions that were full until some moment and then drained are all consistent with “empty now”. The non-uniqueness is the mathematical shadow of the fact that information about the past has been lost.

Norton's dome

In 2003 the philosopher John Norton described a frictionless dome shaped so that a ball placed exactly at the top obeys Newton's laws in the form $r'' = \sqrt{r}$. One solution is that the ball sits at the top for ever. But there are others, in which it sits there for any length of time and then spontaneously rolls off in any direction, without any cause. The example is still debated: is classical mechanics deterministic after all? Mathematically, it is non-uniqueness of exactly the kind in this question.

The thinking skill

Don't assume uniqueness

Finding a solution does not show it is the only one. Ask whether others exist, especially where the equation involves functions that are not smooth.

At university, existence and uniqueness theorems tell you exactly what conditions are needed. Here a square root, with its infinite slope at zero, was enough to break uniqueness.

Try this next

- 1 **Two rates.** (From STEP II, 2018.) Using $v = \sqrt{y}$, solve $\frac{dy}{dt} = \alpha y^{1/2} - \beta y$ (with $\alpha, \beta > 0$) and find the non-constant solution y_1 with $y_1(0) = 0$. Do the same for $\frac{dy}{dt} = \alpha y^{2/3} - \beta y$, using $v = y^{1/3}$. Why does $y = 0$ also satisfy both equations with $y(0) = 0$?
- 2 **The leaking bucket.** Solve $h' = -k\sqrt{h}$ with $h(0) = h_0 > 0$. When is the bucket empty? Write down two different solutions with $h(T) = 0$ for some T and $h(t) = 0$ for all $t \geq T$.
- 3 **Uniqueness restored.** Show that $y' = 2y$ with $y(0) = 0$ has only the solution $y = 0$. (*Hint:* consider $\frac{d}{dx}(ye^{-2x})$.) What property does $2y$ have that $2\sqrt{y}$ lacks?
- 4 **Cubes.** Find all solutions of $y' = 3y^{2/3}$ through the origin.

Further reading

- **Morris Hirsch, Stephen Smale & Robert Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*** (Academic Press). Existence and uniqueness, and much of the rest of this chapter.
- **George Simmons, *Differential Equations with Applications and Historical Notes*** (CRC Press). A classic, full of history; discusses singular solutions and envelopes.
- **John Norton, “The dome: an unexpectedly simple failure of determinism”, *Philosophy of Science* 75** (2008). Short and readable.

8.2 Bounding Without Solving

The question

- (i) Let $y(x)$ be a solution of the differential equation $\frac{d^2y}{dx^2} + y^3 = 0$ with $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^4.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leq 1$ for all x .

- (ii) Let $v(x)$ be a solution of the differential equation $\frac{d^2v}{dx^2} + x\frac{dv}{dx} + \sinh v = 0$ with $v = \ln 3$ and $\frac{dv}{dx} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{dv}{dx}\right)^2 + 2 \cosh v.$$

Show that $\frac{dE}{dx} \leq 0$ for $x \geq 0$ and deduce that $\cosh v(x) \leq \frac{5}{3}$ for $x \geq 0$.

- (iii) Let $w(x)$ be a solution of the differential equation

$$\frac{d^2w}{dx^2} + (5 \cosh x - 4 \sinh x - 3) \frac{dw}{dx} + (w \cosh w + 2 \sinh w) = 0$$

with $\frac{dw}{dx} = \frac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leq \frac{5}{4}$ for $x \geq 0$.

The hook

Most differential equations cannot be solved. There is no formula for the solution of $y'' + y^3 = 0$ in terms of familiar functions, let alone for the monster in part (iii). Yet we can still say a great deal about the solutions: that they stay bounded, that they oscillate, that they settle down. The trick is to find a quantity, built from the solution and its derivative, that never increases. Then the solution is trapped.

In physics, that quantity is usually *energy*. A ball rolling in a bowl cannot climb higher than its starting height, because energy is conserved, or, with friction, decreases. This

question applies the same reasoning to three equations, the last of them chosen so that you have to invent the “energy” yourself. The method, due to Aleksandr Lyapunov in 1892, is one of the most important tools in the study of differential equations.

Before you start

Ask yourself For part (i), think of y as the position of a particle and x as time. Then $y'' = -y^3$ is Newton’s law with force $-y^3$. What are the kinetic and potential energies?

The kinetic energy is $\frac{1}{2}(y')^2$ (taking mass 1), and the force $-y^3$ comes from the potential $V(y) = \frac{1}{4}y^4$, since force is $-V'(y)$. The total energy is $\frac{1}{2}(y')^2 + \frac{1}{4}y^4$, which is half of E . Conservation of energy is exactly the statement that E is constant.

Working it through

STAGE 1 Conservation

By the chain rule,

$$\frac{dE}{dx} = 2y'y'' + 2y^3y' = 2y'(y'' + y^3) = 0.$$

So E is constant, equal to its value at $x = 0$: $E = 0 + \frac{1}{2} = \frac{1}{2}$. Since $(y')^2 \geq 0$, we get $\frac{1}{2}y^4 \leq \frac{1}{2}$, so $y^4 \leq 1$ and $|y| \leq 1$. ■

The solution oscillates between -1 and 1 , tracing the closed curve $(y')^2 + \frac{1}{2}y^4 = \frac{1}{2}$ in the (y, y') plane (Figure 8.2.1).

STAGE 2 Dissipation

Ask yourself Differentiate $E = (v')^2 + 2 \cosh v$. Use the differential equation to replace v'' . What sign does the result have?

$$\frac{dE}{dx} = 2v'v'' + 2 \sinh v v' = 2v'(v'' + \sinh v) = 2v'(-xv') = -2x(v')^2 \leq 0 \quad \text{for } x \geq 0.$$

The term xv' acts like friction, and it removes energy. So $E(x) \leq E(0)$ for $x \geq 0$. At $x = 0$, $\cosh(\ln 3) = \frac{3+1}{2} = \frac{5}{2}$, so $E(0) = \frac{10}{3}$. Hence

$$2 \cosh v \leq (v')^2 + 2 \cosh v = E(x) \leq \frac{10}{3}, \quad \text{so} \quad \cosh v(x) \leq \frac{5}{3}. \quad \blacksquare$$

STAGE 3 Inventing the energy

Part (iii) gives no hint. We need a function $E = (w')^2 + F(w)$ that does not increase.

Ask yourself If $E = (w')^2 + F(w)$, then $E' = 2w'w'' + F'(w)w'$. What should $F'(w)$ be, so that the equation can be used to replace w'' and leave only a multiple of $(w')^2$?

We want $E' = 2w'(w'' + w \cosh w + 2 \sinh w)$, so we need $F'(w) = 2(w \cosh w + 2 \sinh w)$. Since $\frac{d}{dw}(w \sinh w) = \sinh w + w \cosh w$, we can take

$$F(w) = 2w \sinh w + 2 \cosh w, \quad E = (w')^2 + 2w \sinh w + 2 \cosh w.$$

Then, using the equation,

$$\frac{dE}{dx} = -2(5 \cosh x - 4 \sinh x - 3)(w')^2.$$

Pause and try Show that $5 \cosh x - 4 \sinh x - 3 \geq 0$ for all x . (Write it in terms of e^x and look for a perfect square.) Then finish the argument.

$5 \cosh x - 4 \sinh x - 3 = \frac{5}{2}(e^x + e^{-x}) - 2(e^x - e^{-x}) - 3 = \frac{1}{2}e^x - 3 + \frac{9}{2}e^{-x} = \frac{1}{2}(e^{x/2} - 3e^{-x/2})^2 \geq 0$. So E never increases. At $x = 0$, $E(0) = \frac{1}{2} + 0 + 2 = \frac{5}{2}$. Since $w \sinh w \geq 0$ for every w (both factors have the same sign),

$$2 \cosh w \leq (w')^2 + 2w \sinh w + 2 \cosh w = E(x) \leq \frac{5}{2}, \quad \text{so} \quad \cosh w(x) \leq \frac{5}{4}. \quad \blacksquare$$

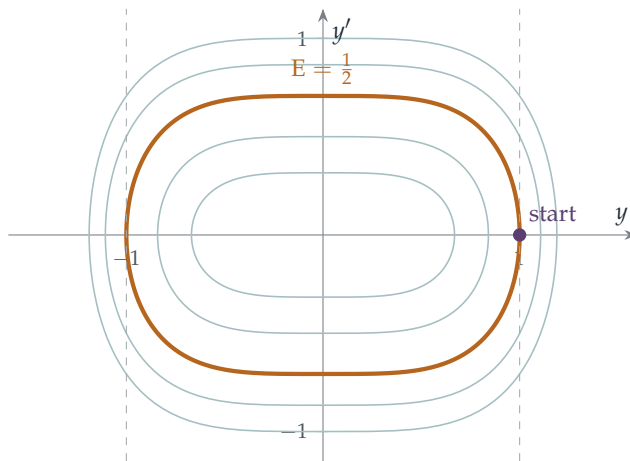


Figure 8.2.1 Level curves of the energy $E = (y')^2 + \frac{1}{2}y^4$ in the (y, y') plane. Because E is constant, the solution of part (i), starting at $y = 1$, $y' = 0$, stays on the curve $E = \frac{1}{2}$ (amber) for ever, and so never leaves the strip $|y| \leq 1$ (dashed). We know this without ever finding a formula for y .

Looking back

► **Multiply by y' .** In each part, E' was $2y'$ times the left-hand side of the equation, plus

whatever was left over. That leftover term decided whether E was conserved or decreasing.

- **Energy bounds position.** Since the kinetic part $(y')^2$ is non-negative, a bound on E is a bound on the potential part, and hence on y .
- **Spot hidden squares.** The damping coefficient in part (iii) looked as if it might change sign, but it was a perfect square in disguise.

What's really going on

Lyapunov functions

In his 1892 doctoral thesis, Aleksandr Lyapunov studied the *stability* of motion: if a system is disturbed slightly from equilibrium, does it return, or wander away? His “second method” was to find a function, now called a *Lyapunov function*, that is smallest at the equilibrium and never increases along solutions. Then solutions that start near the equilibrium stay near it. The beauty of the method is that it needs no formula for the solutions, only the differential equation itself, which is exactly what this question demonstrates. Lyapunov functions are now used to prove the stability of everything from aircraft control systems to power grids to ecosystems.

Energy in physics and engineering

Part (i) is a nonlinear oscillator: like a spring, but with a restoring force proportional to the cube of the displacement. Unlike a simple harmonic oscillator, its period depends on its amplitude (larger oscillations are faster), but energy conservation still confines it. Part (ii) adds damping that grows with time, and energy drains away. In engineering, energy methods are used to show that structures will not shake themselves apart and that control systems will not oscillate out of control. In the theory of partial differential equations, “energy estimates” of this kind are the main tool for proving that solutions exist and behave well; questions about the equations of fluid flow, including one of the Millennium Prize Problems (the Navier–Stokes problem), come down to whether such bounds can be found.

The thinking skill

Find a quantity that cannot increase

If you cannot solve an equation, look for a quantity built from the solution whose derivative has a definite sign. It will bound the solution.

The search is guided by physics (“what is the energy?”) and by algebra (“what must I add so that the derivative collapses?”). Part (iii) shows the algebraic route

when there is no physical intuition to lean on.

Try this next

- 1 **Simple harmonic motion.** For $y'' + y = 0$ with $y(0) = A$, $y'(0) = 0$, find the conserved energy and deduce $|y| \leq A$. Check against the exact solution.
- 2 **The pendulum.** The angle θ of a pendulum satisfies $\theta'' + \sin \theta = 0$ (in suitable units). Show that $(\theta')^2 - 2 \cos \theta$ is constant. If the pendulum starts at rest at angle θ_0 , show that it never exceeds θ_0 in absolute value.
- 3 **Damping.** For $y'' + 2ky' + y = 0$ with $k > 0$, show that $(y')^2 + y^2$ decreases. Does it necessarily tend to zero?
- 4 **An escape.** For $y'' - y^3 = 0$ with $y(0) = 1$, $y'(0) = 0$, find a conserved quantity and show that y is not bounded by it. What goes wrong?

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapters 6 and 7 cover phase planes, conservative systems and Lyapunov functions.
- **D. W. Jordan & P. Smith, *Nonlinear Ordinary Differential Equations*** (Oxford University Press). A standard undergraduate text with many examples of energy methods.
- **Keith Devlin, *The Millennium Problems*** (Basic Books). Includes an accessible account of the Navier–Stokes problem.

8.3 Harvesting a Population

The question

If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

$$x(t) = K + K \frac{1 - e^{-2Kt}}{1 + e^{-2Kt}}.$$

Sketch, for $t \geq 0$, the graph of x against t . What happens to $x(t)$ as $t \rightarrow \infty$?

Now suppose that the situation is as described in the first paragraph, except that we remove the bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x . By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \rightarrow K + \alpha$ as $t \rightarrow \infty$.

The hook

A population with unlimited food grows exponentially. Real populations run out of room: growth slows as the population approaches what the environment can support, its *carrying capacity*. The simplest model of this, the *logistic equation*, is in the first half of the question, and it describes everything from bacteria in a dish to the spread of a new technology.

The second half asks a question with enormous practical stakes: what happens if we harvest the population at a steady rate? Below a certain threshold, the population adjusts to a new, lower equilibrium. But push the harvest past that threshold, even slightly, and the equilibrium vanishes and the population crashes to zero. This is the mathematics behind the collapse of fisheries, and the reason fishing quotas exist.

Before you start



Ask yourself Before solving anything, sketch the growth rate $(2K - x)x$ as a function of x . Where is it zero? Where positive? What does that tell you about how x will behave?

The rate is zero at $x = 0$ and $x = 2K$, positive in between and negative beyond $2K$. So a population starting between 0 and $2K$ grows towards $2K$, and one starting above $2K$

shrinks towards it. The carrying capacity is $2K$, and the growth is fastest at $x = K$, where the rate is K^2 . You can predict the long-term behaviour without solving the equation.

Working it through

STAGE 1 Solving the logistic equation

Ask yourself Separate the variables: $\int \frac{dx}{(2K-x)x} = \int dt$. Split the left-hand side into partial fractions.

$$\frac{1}{(2K-x)x} = \frac{1}{2K} \left(\frac{1}{x} + \frac{1}{2K-x} \right), \text{ so}$$

$$\frac{1}{2K} \ln \frac{x}{2K-x} = t + C.$$

At $t = 0$, $x = K$, so $\ln 1 = 0$ and $C = 0$. Then $\frac{x}{2K-x} = e^{2Kt}$, giving

$$x = \frac{2Ke^{2Kt}}{1 + e^{2Kt}} = \frac{2K}{1 + e^{-2Kt}} = K + K \frac{1 - e^{-2Kt}}{1 + e^{-2Kt}}. \quad \blacksquare$$

The fraction is $\tanh(Kt)$, so $x = K + K \tanh Kt$: an S-shaped curve rising from K towards $2K$. As $t \rightarrow \infty$, $e^{-2Kt} \rightarrow 0$ and $x \rightarrow 2K$.

STAGE 2 Harvesting

Removing bacteria at rate L gives

$$\frac{dx}{dt} = (2K-x)x - L.$$

Ask yourself Complete the square in $(2K-x)x - L$. Where are its zeros?

$(2K-x)x - L = K^2 - L - (x-K)^2 = \alpha^2 - (x-K)^2 = (\alpha - (x-K))(\alpha + (x-K))$. The equilibria are $x = K \pm \alpha$.

Pause and try Now substitute $y = x - K + \alpha$. What equation does y satisfy? Compare it with the original logistic equation.

With $y = x - K + \alpha$, we have $x - K = y - \alpha$, so $\alpha + (x - K) = y$ and $\alpha - (x - K) = 2\alpha - y$. Hence

$$\frac{dy}{dt} = (2\alpha - y)y,$$

the logistic equation again, with K replaced by α . Initially $y(0) = K - K + \alpha = \alpha$, exactly as $x(0) = K$ in the first part. So by Stage 1 (with α for K), $y(t) = \alpha + \alpha \tanh \alpha t \rightarrow 2\alpha$, and therefore

$$x(t) = K + \alpha \tanh \alpha t \rightarrow K + \alpha. \quad \blacksquare$$

Looking back

- **Qualitative first.** The sign of the growth rate predicted the long-term behaviour before any solving.
- **Completing the square reveals the equilibria.** The harvested equation had equilibria at $K \pm \alpha$.
- **A substitution reduces a new problem to an old one.** Shifting the variable turned the harvested equation into the logistic equation already solved.

What's really going on

Tipping points

The harvested equation has two equilibria, $K + \alpha$ and $K - \alpha$. The upper one is *stable*: a population slightly above or below it returns to it. The lower one is *unstable*: a population just below $K - \alpha$ declines to extinction. As the harvest rate L increases, $\alpha = \sqrt{K^2 - L}$ shrinks, and the two equilibria approach each other; at $L = K^2$ they merge, and for $L > K^2$ there is no equilibrium at all, and every population collapses (Figure 8.3.1). This is a *saddle-node bifurcation*, the simplest kind of *tipping point*. Its sting is that nothing dramatic warns of it: as L approaches K^2 , the equilibrium population declines only gently, and then suddenly there is no equilibrium. Reducing the harvest afterwards may not help, because the population may already be below the unstable threshold.

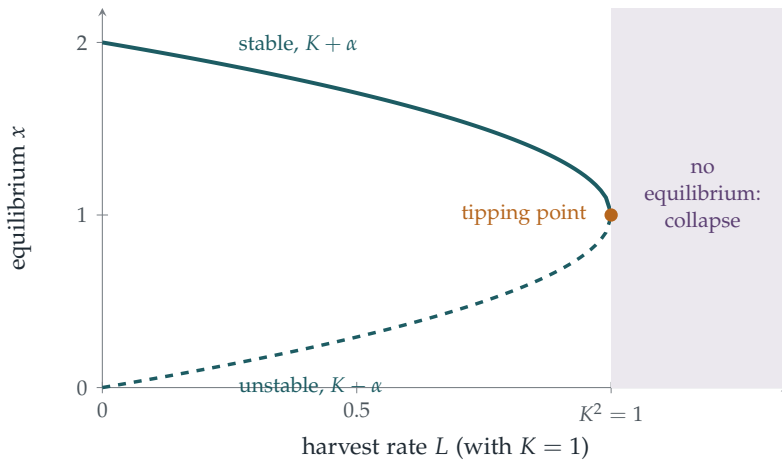


Figure 8.3.1 Equilibrium populations against the harvest rate L , with $K = 1$. The stable equilibrium (solid) and the unstable threshold (dashed) approach each other as L increases, and meet at $L = K^2$. Beyond that, no equilibrium exists and the population collapses whatever its size.

Maximum sustainable yield

What is the largest harvest the population can sustain? It is $L = K^2$, the peak of the growth rate $(2K - x)x$, achieved when the population is held at K , half the carrying capacity. Fisheries science adopted this “maximum sustainable yield” as a management target in the mid-twentieth century. But harvesting exactly at the maximum puts the system exactly at the tipping point, where the slightest overestimate of K , or a bad year, causes collapse. The Grand Banks cod fishery off Newfoundland, one of the richest in the world for five hundred years, collapsed in the early 1990s, and in 1992 the Canadian government imposed a moratorium that put tens of thousands of people out of work. The stocks have still not fully recovered.

Logistic growth elsewhere

The logistic equation was proposed by Pierre-François Verhulst in 1838 to model human populations. The same S-shaped curve describes the spread of an epidemic in its early stages, the adoption of new technologies, autocatalytic chemical reactions and the growth of tumours. Its discrete-time version, the logistic *map*, behaves very differently: it can be chaotic, as we will see in Section 8.7.

The thinking skill

Equilibria and their stability

For an equation $x' = f(x)$, the equilibria are the zeros of f , and the sign of f between them tells you which way solutions move.

An equilibrium where f changes from positive to negative is stable; one where it changes from negative to positive is unstable. Drawing the graph of f often answers the important questions without solving anything.

Try this next

- 1 **Starting low.** With harvesting and $K^2 > L$, show that if $x(0) < K - \alpha$ then the population reaches zero in finite time. (*Hint:* what does y do when $y(0) < 0$?)
- 2 **Over-harvesting.** If $L > K^2$, write $\beta = \sqrt{L - K^2}$. Solve $x' = -(x - K)^2 - \beta^2$ with $x(0) = 2K$ and find when the population reaches zero.
- 3 **Proportional harvesting.** Suppose instead that a fixed *fraction* of the population is removed: $x' = (2K - x)x - hx$. Find the equilibria and show that the population survives for all $h < 2K$. Why is this policy safer?
- 4 **Half-time.** In the unharvested case, find the time at which the population reaches $\frac{3}{2}K$.

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapter 3 is about bifurcations, with this harvesting model as an exercise.
- **Nicholas Britton, *Essential Mathematical Biology*** (Springer). Population models for undergraduates.
- **Mark Kurlansky, *Cod: A Biography of the Fish That Changed the World*** (Penguin). The history of the cod fishery and its collapse.

8.4 Spiralling onto a Circle

The question

- (i) Show that under the changes of variable $x = r \cos \theta$ and $y = r \sin \theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y + x) \frac{dy}{dx} = y - x$$

becomes

$$\frac{dr}{d\theta} + r = 0.$$

Sketch a solution in the x - y plane.

- (ii) Show that the solutions of

$$(y + x - x(x^2 + y^2)) \frac{dy}{dx} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + Ae^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A .

The hook

Many systems in nature settle into a regular rhythm that does not depend on how they started: a heart beating, a cricket chirping, a violin string bowed steadily, the population cycles of predators and prey. Disturb them a little and they return to the same rhythm. In the language of differential equations, their solutions approach a closed loop, called a *limit cycle*.

This question constructs one. The equation looks hopeless in x and y , but in polar coordinates it collapses to something you can solve, and the solutions turn out to spiral onto the unit circle, from inside and from outside. The key move, choosing coordinates that match the symmetry of the problem, is one we have used throughout the book.

Before you start



Ask yourself Both equations contain the combinations $x dx + y dy$ and $x dy - y dx$ in disguise. What are these in polar coordinates?

Since $r^2 = x^2 + y^2$, differentiating gives $x dx + y dy = r dr$. And $x dy - y dx = r^2 d\theta$ (this is the area formula from Section 4.9). These two identities make the change of variables almost painless: rewrite the equation so that only these combinations appear.

Working it through

STAGE 1 Part (i) in polar coordinates

Write the equation as $(y + x) dy - (y - x) dx = 0$, and rearrange:

$$(x dx + y dy) + (x dy - y dx) = 0.$$



Ask yourself Check that rearrangement by expanding. Now use the two identities above.

$(y + x) dy - (y - x) dx = y dy + x dy - y dx + x dx$, which is the expression above. In polar form it is $r dr + r^2 d\theta = 0$, and dividing by $r^2 d\theta$ (with $r > 0$),

$$\frac{dr}{d\theta} + r = 0. \quad \blacksquare$$

The solutions are $r = Ce^{-\theta}$: *logarithmic spirals*, winding in towards the origin as θ increases and out to infinity as θ decreases.

STAGE 2 Part (ii)



Pause and try The new equation is the old one plus extra terms involving $x^2 + y^2 = r^2$. Rewrite it in the same way and find $\frac{dr}{d\theta}$.

The equation is $(y + x - xr^2) dy - (y - x - yr^2) dx = 0$, that is,

$$[(y + x) dy - (y - x) dx] - r^2(x dy - y dx) = 0.$$

The first bracket is $r dr + r^2 d\theta$ as before, and the second is $r^2 d\theta$. So $r dr + r^2(1 - r^2) d\theta = 0$, giving

$$\frac{dr}{d\theta} = r^3 - r.$$



Ask yourself This is not linear in r , but it becomes linear in $u = r^{-2}$. What equation does u satisfy?

With $u = r^{-2}$, $\frac{du}{d\theta} = -2r^{-3} \frac{dr}{d\theta} = -2r^{-3}(r^3 - r) = -2 + 2u$. So $\frac{du}{d\theta} - 2u = -2$, whose solutions are $u = 1 + Ae^{2\theta}$. Hence

$$r^2 = \frac{1}{1 + Ae^{2\theta}}. \quad \blacksquare$$

STAGE 3 The three kinds of solution

- $A = 0$: $r = 1$. The unit circle is a solution.
- $A > 0$: $r < 1$ always. As $\theta \rightarrow -\infty$, $r \rightarrow 1$; as $\theta \rightarrow +\infty$, $r \rightarrow 0$. The curve spirals between the circle and the origin.
- $A < 0$: $r > 1$, and the formula only makes sense while $1 + Ae^{2\theta} > 0$, that is, $\theta < \frac{1}{2} \ln\left(-\frac{1}{A}\right)$. As $\theta \rightarrow -\infty$, $r \rightarrow 1$; as θ approaches that bound, $r \rightarrow \infty$. The curve spirals between the circle and infinity.

So every solution curve approaches the unit circle as $\theta \rightarrow -\infty$, from inside or outside (Figure 8.4.1).

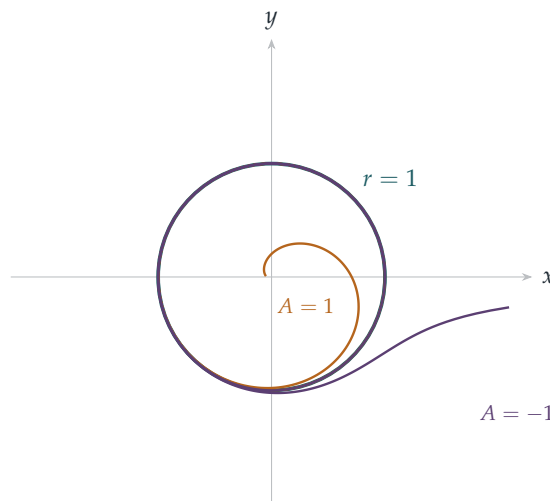


Figure 8.4.1 Solutions of part (ii). The unit circle (teal) is a solution. The solution with $A = 1$ (amber) lies inside it, spiralling from the circle (as $\theta \rightarrow -\infty$) towards the origin; the solution with $A = -1$ (plum) lies outside, spiralling from the circle out to infinity. Seen in the other direction (decreasing θ), every solution winds onto the circle: a limit cycle.

Looking back

- **Use the symmetry.** The equations treated all directions alike, and polar coordinates made that visible.
- **Recognise standard combinations.** $x \, dx + y \, dy = r \, dr$ and $x \, dy - y \, dx = r^2 \, d\theta$ did most of the work.
- **A substitution linearises.** The nonlinear equation $r' = r^3 - r$ became linear in $u = r^{-2}$ (a *Bernoulli equation*).

What's really going on

Limit cycles

Henri Poincaré introduced limit cycles in the 1880s while studying the qualitative behaviour of differential equations in the plane. Interpret the question's equation as describing the path of a point moving in the plane. The unit circle is an isolated closed orbit, and every nearby orbit winds towards it (or away from it, depending on the direction of time). In 1926 the Dutch engineer Balthasar van der Pol found a limit cycle in an electrical circuit with a valve, and proposed that similar equations describe the beating of the heart. Limit cycles model any system with a self-sustained, stable rhythm: circadian clocks, nerve impulses, the cycles of predator and prey populations.

No chaos in the plane

The *Poincaré–Bendixson theorem* says that for a smooth system in the plane, an orbit that stays in a bounded region without approaching an equilibrium must approach a closed orbit. So in two dimensions, long-term behaviour is limited to equilibria and cycles: there is no room for chaos, because orbits cannot cross. Chaos needs at least three dimensions, as Edward Lorenz discovered in 1963 with his famous three-variable model of convection (the “butterfly”). In discrete time, as we will see in Section 8.7, chaos is possible even in one dimension.

An unsolved problem

In 1900 David Hilbert asked, as part of his sixteenth problem: what is the largest number of limit cycles that a system $x' = P(x, y)$, $y' = Q(x, y)$ can have, when P and Q are polynomials of degree n ? It is still unknown even for $n = 2$, where systems with four limit cycles are known and it is not known whether more are possible. It is one of the very few of Hilbert's problems that remain completely open.

The thinking skill

Change coordinates to match the symmetry

If a problem has rotational symmetry, use polar coordinates; if it has another symmetry, find coordinates that respect it.

The right coordinates turned an intractable equation into a separable one, and made the geometry of the solutions (circles and spirals) obvious.

Try this next

- 1 **The spiral.** For the solution $r = e^{-\theta}$ of part (i), show that the curve crosses every ray from the origin at the same angle, 45° . (This is why logarithmic spirals are called *equiangular*.)
- 2 **Polar identities.** Prove that $x \, dx + y \, dy = r \, dr$ and $x \, dy - y \, dx = r^2 \, d\theta$.
- 3 **A system in time.** Consider the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$. Show that every solution with $r(0) > 0$ approaches the unit circle as $t \rightarrow \infty$. Write the system in terms of x and y .
- 4 **Two cycles.** Find a system of the form $\dot{r} = f(r)$, $\dot{\theta} = 1$ with two limit cycles, at $r = 1$ and $r = 2$, one attracting and one repelling.

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapter 7 is about limit cycles and the Poincaré–Bendixson theorem.
- **Steven Strogatz, *Sync*** (Hyperion). A popular book about rhythms and synchronisation in nature.
- **James Gleick, *Chaos: Making a New Science*** (Penguin). The story of Lorenz and the discovery of chaos.

8.5 Greyhounds Chasing Greyhounds

The question

Four greyhounds A , B , C and D are held at positions such that $ABCD$ is a large square. At a given instant, the dogs are released and A runs directly towards B at constant speed v , B runs directly towards C at constant speed v , and so on. Show that A 's path is given in polar coordinates (referred to an origin at the centre of the field and a suitable initial line) by $r = \lambda e^{-\theta}$, where λ is a constant.

Generalise this result to the case of n dogs held at the vertices of a regular n -gon ($n \geq 3$).

The hook

Four dogs stand at the corners of a square field. Each chases the dog to its left. Where do they meet, how long does it take, and what paths do they follow? The paths are beautiful spirals that wind round the centre infinitely many times, and yet each dog runs only a finite distance, exactly the length of a side of the square.

This is a *pursuit problem*, a type first studied by Pierre Bouguer in 1732 for a ship chasing a pirate. The trick that makes it solvable is symmetry: whatever happens to one dog happens to all of them, so the four dogs always form a square. Once you see that, the differential equation almost writes itself.

Before you start



Ask yourself By symmetry, what shape do the four dogs form at every moment? Where is its centre?

The situation is unchanged by a quarter-turn about the centre of the square (which moves $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$), so the dogs always form a square with the same centre O . The square rotates and shrinks, but it stays a square. In particular, A and B are always the same distance r from O , and OB is always a quarter-turn ahead of OA .

Working it through

STAGE 1 The direction of motion

Ask yourself A runs towards B . In the triangle OAB , what angle does AB make with AO (the direction towards the centre)?

Triangle OAB has $OA = OB$ and a right angle at O , so its other two angles are 45° . So A 's velocity, of size v , points at 45° to the inward radial direction, and is turned towards B , in the direction of increasing θ (if we label the angles so that B is anticlockwise from A).

STAGE 2 The differential equation

Ask yourself Split A 's velocity into radial and transverse parts. In polar coordinates the radial velocity is $\dot{r}$ and the transverse velocity is $r\dot{\theta}$.

The radial component (inwards) is $v \cos 45^\circ$ and the transverse component is $v \sin 45^\circ$, so

$$\dot{r} = -\frac{v}{\sqrt{2}}, \quad r\dot{\theta} = \frac{v}{\sqrt{2}}.$$

Dividing, $\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = -r$, so $r = \lambda e^{-\theta}$, with λ the initial distance from the centre if $\theta = 0$ initially. ■

STAGE 3 n dogs

Pause and try For a regular n -gon, what is the angle at O in triangle OAB ? What angle does AB make with AO ? Repeat Stage 2.

The angle at O is $\frac{2\pi}{n}$, so the two equal base angles are $\frac{1}{2}(\pi - \frac{2\pi}{n}) = \frac{\pi}{2} - \frac{\pi}{n}$. The velocity makes this angle with the inward radius, so

$$\dot{r} = -v \cos\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = -v \sin \frac{\pi}{n}, \quad r\dot{\theta} = v \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = v \cos \frac{\pi}{n}.$$

Hence $\frac{dr}{d\theta} = -r \tan \frac{\pi}{n}$, and

$$r = \lambda e^{-\theta \tan(\pi/n)}. \quad \blacksquare$$

For $n = 4$, $\tan \frac{\pi}{4} = 1$, recovering the square case.

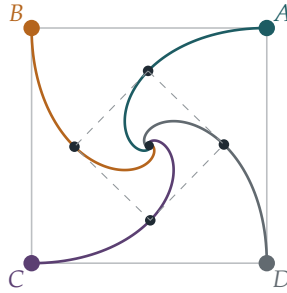


Figure 8.5.1 Four dogs chasing each other from the corners of a square. Each follows a logarithmic spiral $r = \lambda e^{-\theta}$ (relative to its own starting direction), winding infinitely often round the centre. At every moment the dogs form a square (dashed, shown at one intermediate time), rotating and shrinking until they meet at the centre.

Looking back

- **Symmetry reduces many bodies to one.** Four interacting dogs became a single particle obeying a simple rule, because the configuration stays a square.
- **Polar coordinates for motion about a centre.** Radial and transverse components of velocity, $\dot{r}$ and $r\dot{\theta}$, gave the equation directly.
- **Divide to eliminate time.** The ratio $\dot{r}/\dot{\theta}$ gave the shape of the path without needing to know when each point is reached.

What's really going on

Infinitely many turns, finite distance

The radial speed $\dot{r} = -v \sin \frac{\pi}{n}$ is constant, so the dogs reach the centre after time $T = \frac{r_0}{v \sin(\pi/n)}$, having each run a distance $vT = \frac{r_0}{\sin(\pi/n)}$. For a regular n -gon of side s , $r_0 = \frac{s}{2 \sin(\pi/n)}$, so each dog runs

$$\frac{s}{2 \sin^2(\pi/n)}.$$

For four dogs this is exactly s : each dog runs precisely the length of one side. (Here is a slick way to see it: B 's velocity is always perpendicular to AB , so it does not change the distance AB ; only A 's motion, at speed v straight along AB , closes the gap, and the gap is initially s .) For three dogs the distance is $\frac{2}{3}s$, and for six it is $2s$. Yet in every case the spiral winds round the centre infinitely many times, because $\theta \rightarrow \infty$ as $r \rightarrow 0$. A curve of finite length with infinitely many turns is no paradox: the turns get smaller fast enough.

The logarithmic spiral

The curve $r = \lambda e^{-k\theta}$ is the *logarithmic* or *equiangular* spiral: it crosses every ray from the centre at the same angle. It appears in nature, in the shell of the nautilus and the arms of some galaxies, because growth that preserves shape produces it. It fascinated Jacob Bernoulli so much that he asked for it to be carved on his tombstone with the motto *Eadem mutata resurgo* (“Changed, I shall arise the same”), referring to the fact that it looks the same at every scale. The stonemason carved an Archimedean spiral instead.

Pursuit in practice

“Pure pursuit”, always heading straight at the target, is the simplest guidance law for a missile or a predator. It is not the best: heading for where the target *will* be (“proportional navigation”) is more efficient, and is what both modern missiles and some predators, such as dragonflies and falcons, actually do. The mathematics of pursuit and evasion became a subject of its own in the twentieth century under the name *differential games*.

The thinking skill

Exploit symmetry to reduce a many-body problem

If a system is symmetric and the rules respect the symmetry, the symmetry is preserved for all time, and you can follow just one part.

This idea solves many problems in mechanics and dynamics; it is a simple case of the deep connection between symmetry and conservation.

Try this next

- 1 The slick argument.** Explain carefully why the distance AB decreases at rate exactly v in the four-dog problem, and deduce that the dogs meet after time s/v .
- 2 Three dogs.** Three dogs start at the corners of an equilateral triangle of side s . Find the equation of each path, the time until they meet, and the distance each runs.
- 3 How many turns?** For four dogs starting 100 m from the centre, how close to the centre are they after one full turn? After how many full turns are they within 1 mm?
- 4 Two dogs.** What happens with $n = 2$ (two dogs facing each other)? Does the formula still make sense?

Further reading

- ▶ **Paul Nahin, *Chases and Escapes: The Mathematics of Pursuit and Evasion*** (Princeton University Press). Everything about pursuit curves, from Bouguer to missiles.
- ▶ **Eli Maor, *e: The Story of a Number*** (Princeton University Press). Includes the logarithmic spiral and Bernoulli's tombstone.
- ▶ **D'Arcy Thompson, *On Growth and Form*** (Cambridge University Press). The classic book on mathematics in nature, including spiral growth.

8.6 The Quantum Oscillator

The question

For $n = 0, 1, 2, \dots$, the functions y_n satisfy the differential equation

$$\frac{d^2 y_n}{dx^2} - \omega^2 x^2 y_n = -(2n + 1)\omega y_n,$$

where ω is a positive constant, and $y_n \rightarrow 0$ and $dy_n/dx \rightarrow 0$ as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = e^{-\lambda x^2} \quad \text{and} \quad y_1(x) = x e^{-\lambda x^2}$$

for some constant λ , to be determined.

Show that

$$\frac{d}{dx} \left(y_m \frac{dy_n}{dx} - y_n \frac{dy_m}{dx} \right) = 2(m - n)\omega y_m y_n,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x) dx = 0.$$

The hook

A mass on a spring oscillates with any energy you like: pull it back a little or a lot. But in 1900 Max Planck found that he could only explain the colours of light emitted by hot objects if the vibrating atoms could have only certain energies, in steps of a fixed size. Energy, at the atomic scale, comes in *quanta*. Twenty-five years later, quantum mechanics explained why.

This question contains the explanation. The differential equation is, after a change of units, the Schrödinger equation for a quantum particle in a spring-like potential: the *quantum harmonic oscillator*. It has well-behaved solutions, ones that die away at infinity, only for special values of the energy, labelled by $n = 0, 1, 2, \dots$, in equally spaced steps. And the solutions for different n are *orthogonal*: the integral of their product is zero, just like $\sin mx$ and $\sin nx$ in a Fourier series. The quantum oscillator is the starting point for the physics of molecules, solids and light.

Before you start

? **Ask yourself** Why should a function like $e^{-\lambda x^2}$ be a sensible guess? What happens to $e^{-\lambda x^2}$ and its derivative as $x \rightarrow \pm\infty$?

For large $|x|$ the equation is dominated by $y'' \approx \omega^2 x^2 y$, and a Gaussian $e^{-\lambda x^2}$ has $y'' \approx 4\lambda^2 x^2 y$ for large x , which matches if $4\lambda^2 = \omega^2$. With $\lambda > 0$, a Gaussian and all its derivatives decay to 0 faster than any power of x . So the boundary conditions will be automatic; the work is in matching the equation.

Working it through

STAGE 1 The ground state

? **Ask yourself** Differentiate $y_0 = e^{-\lambda x^2}$ twice, substitute into the equation with $n = 0$, and compare coefficients of x^2 and of the constant term.

$y'_0 = -2\lambda x e^{-\lambda x^2}$ and $y''_0 = (4\lambda^2 x^2 - 2\lambda)e^{-\lambda x^2}$. The equation with $n = 0$ requires

$$4\lambda^2 x^2 - 2\lambda - \omega^2 x^2 = -\omega$$

for all x . Matching coefficients: $4\lambda^2 = \omega^2$ and $2\lambda = \omega$. Both hold with $\lambda = \frac{\omega}{2}$ (taking $\lambda > 0$, so that $y_0 \rightarrow 0$ as $x \rightarrow \pm\infty$). ■

STAGE 2 The first excited state

$y_1 = x e^{-\lambda x^2}$ has $y'_1 = (1 - 2\lambda x^2)e^{-\lambda x^2}$ and $y''_1 = (4\lambda^2 x^3 - 6\lambda x)e^{-\lambda x^2}$. The equation with $n = 1$ requires

$$4\lambda^2 x^3 - 6\lambda x - \omega^2 x^3 = -3\omega x,$$

which holds with the same $\lambda = \frac{\omega}{2}$. Again y_1 and y'_1 decay at $\pm\infty$. ■

STAGE 3 An identity

? **Ask yourself** Differentiate $y_m y'_n - y_n y'_m$ by the product rule. Two of the four terms cancel. Then use the differential equation to replace y''_n and y''_m .

$\frac{d}{dx}(y_m y'_n - y_n y'_m) = y'_m y'_n + y_m y''_n - y'_n y'_m - y_n y''_m = y_m y''_n - y_n y''_m$. From the equation, $y''_n = (\omega^2 x^2 - (2n+1)\omega)y_n$, and similarly for m . So

$$y_m y''_n - y_n y''_m = y_m y_n [\omega^2 x^2 - (2n+1)\omega - \omega^2 x^2 + (2m+1)\omega] = 2(m-n)\omega y_m y_n. \quad \blacksquare$$

STAGE 4 Orthogonality

 **Pause and try** Integrate the identity from $-\infty$ to ∞ . What is the left-hand side?

The left-hand side integrates to $[y_m y'_n - y_n y'_m]_{-\infty}^{\infty} = 0 - 0 = 0$, because each y and y' tends to 0 at both ends. So $2(m - n)\omega \int_{-\infty}^{\infty} y_m y_n dx = 0$, and since $m \neq n$ and $\omega > 0$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x) dx = 0. \quad \blacksquare$$

For y_0 and y_1 this is easy to see directly: $y_0 y_1 = x e^{-\omega x^2}$ is an odd function. But the identity proves it for every pair, without knowing the functions.

Looking back

- **Match coefficients.** Substituting a guessed form and requiring the equation to hold for all x determined λ .
- **The Wronskian trick.** The combination $y_m y'_n - y_n y'_m$ has a derivative in which the second derivatives can be replaced using the equation, leaving something simple.
- **Boundary conditions kill boundary terms.** Decay at infinity made the integrated left-hand side vanish, which gave orthogonality.

What's really going on

Schrödinger's equation

In 1926 Erwin Schrödinger proposed that a quantum particle is described by a *wave function* $\psi(x)$, which for a particle of energy E in a potential $V(x)$ satisfies

$$-\frac{\hbar^2}{2m} \psi'' + V(x) \psi = E \psi.$$

For a spring, $V(x) = \frac{1}{2} m \Omega^2 x^2$, and after rescaling x this is exactly the equation in the question, with $E = (n + \frac{1}{2}) \hbar \Omega$. The physical requirement that ψ dies away at infinity (so that the particle is somewhere) can only be met for these energies: that is *quantisation*, falling out of the mathematics. Notice that the lowest energy is not zero but $\frac{1}{2} \hbar \Omega$: a quantum oscillator can never be completely at rest. This *zero-point energy* is real and measurable; it is why liquid helium does not freeze at ordinary pressure, even at absolute zero.

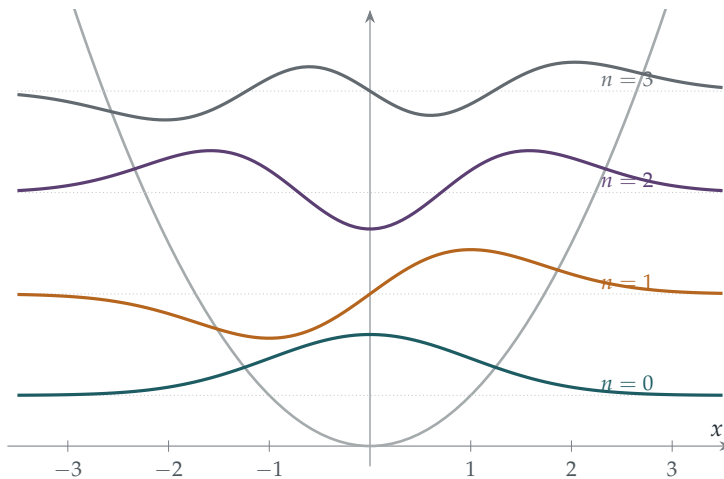


Figure 8.6.1 The quantum harmonic oscillator (with $\omega = 1$). The parabola is the potential energy. Each wave function y_n is drawn at the height of its energy level, $n + \frac{1}{2}$; the levels are equally spaced, and the lowest is above zero. The function y_n crosses zero n times, and any two of them are orthogonal.

Hermite functions and Sturm–Liouville theory

The general solution is $y_n = H_n(\sqrt{\omega}x)e^{-\omega x^2/2}$, where H_n is a polynomial of degree n , the *Hermite polynomial* (Figure 8.6.1). Orthogonality is not special to this equation. Whenever an equation can be written as $(py')' + qy = -\lambda wy$ with suitable boundary conditions, solutions belonging to different values of λ are orthogonal, by exactly the argument of Stage 4. This is *Sturm–Liouville theory* (1830s), and it covers the sine functions of Fourier series (Section 3.7), the vibrations of a drum, the Legendre polynomials of electrostatics and the wave functions of the hydrogen atom. The eigenvalue problem of Section 4.7 is another instance: the critical values found there are the eigenvalues λ , and the orthogonal functions are the natural “modes” of the system.

The thinking skill

Orthogonality from an identity

To show two solutions of related equations are orthogonal, find an exact derivative whose integral vanishes by the boundary conditions, and which equals a multiple of their product.

This is far more powerful than computing the integral directly: it works for every pair at once, even when you cannot write the solutions down.

Try this next

- 1 **The next state.** Show that $y_2 = (4\lambda x^2 - 1)e^{-\lambda x^2}$, with $\lambda = \frac{\omega}{2}$, satisfies the equation with $n = 2$. Check directly that $\int_{-\infty}^{\infty} y_0 y_2 \, dx = 0$, using $\int_{-\infty}^{\infty} e^{-\omega x^2} \, dx = \sqrt{\pi/\omega}$ and $\int_{-\infty}^{\infty} x^2 e^{-\omega x^2} \, dx = \frac{1}{2\omega} \sqrt{\pi/\omega}$.
- 2 **No other energies.** Try $y = e^{-\lambda x^2}$ with n replaced by a general number ν . Show that it works only if $\nu = 0$. (In fact, decaying solutions exist only when ν is a non-negative integer.)
- 3 **Sine waves.** Use the same method to show that $\int_0^\pi \sin mx \sin nx \, dx = 0$ for $m \neq n$, starting from $y'' = -n^2 y$ with $y(0) = y(\pi) = 0$.
- 4 **A ladder.** Show that if y_n solves the equation, then $y_{n+1} = \omega x y_n - y'_n$ solves it with $n + 1$ in place of n . Use this to generate y_1 and y_2 from y_0 . (This is the “raising operator” of quantum mechanics.)

Further reading

- **David Griffiths**, *Introduction to Quantum Mechanics* (Cambridge University Press). The standard undergraduate physics text; Chapter 2 treats the harmonic oscillator.
- **Richard Feynman**, *QED: The Strange Theory of Light and Matter* (Princeton University Press). A popular account of quantum mechanics by one of its masters.
- **Mary Boas**, *Mathematical Methods in the Physical Sciences* (Wiley). Covers Hermite polynomials and Sturm–Liouville theory for students of mathematics and physics.

8.7 Chaos You Can Solve

The question

- (i) The sequence of numbers $u_0, u_1, \dots$ is given by $u_0 = u$ and, for $n \geq 0$,

$$u_{n+1} = 4u_n(1 - u_n). \quad (*)$$

In the case $u = \sin^2 \theta$ for some given angle θ , write down and simplify expressions for u_1 and u_2 in terms of θ . Conjecture an expression for u_n and prove your conjecture.

- (ii) The sequence of numbers $v_0, v_1, \dots$ is given by $v_0 = v$ and, for $n \geq 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geq 0$, $v_{n+1} = -v_n^2 + 2v_n + 2$.

The hook

The rule $u_{n+1} = ku_n(1 - u_n)$ is the *logistic map*, the simplest model of a population with limited resources: u_n is the population in year n as a fraction of the maximum the environment can support, k is the breeding rate, and the factor $1 - u_n$ represents overcrowding. In 1976 the biologist Robert May published a short article in *Nature* called "Simple mathematical models with very complicated dynamics". He showed that, as k increases towards 4, this one-line rule goes from settling down, to oscillating, to behaving in a way that looks completely random. It became one of the founding examples of *chaos*.

Chaotic systems are usually impossible to solve: there is no formula for u_n , and the only way to find u_{100} is to compute all the terms before it. This question finds the exception. At $k = 4$, the most chaotic value of all, there *is* a formula, and it explains exactly why the sequence is unpredictable.

Before you start



Ask yourself Where have you seen the expression $4s(1-s)$ before, with $s = \sin^2 \theta$? Which identity connects $\sin^2 \theta$, $\cos^2 \theta$ and $\sin 2\theta$?

If $s = \sin^2 \theta$ then $1 - s = \cos^2 \theta$, so $4s(1-s) = 4 \sin^2 \theta \cos^2 \theta = (2 \sin \theta \cos \theta)^2 = \sin^2 2\theta$. The map $s \mapsto 4s(1-s)$ is the double-angle formula in disguise.

Working it through

STAGE 1 Guessing the pattern

With $u_0 = \sin^2 \theta$, the identity above gives

$$u_1 = 4 \sin^2 \theta \cos^2 \theta = \sin^2 2\theta, \quad u_2 = 4 \sin^2 2\theta \cos^2 2\theta = \sin^2 4\theta.$$



Ask yourself What happens to the angle at each step? What is your conjecture for u_n ?

The angle doubles each time, so we conjecture $u_n = \sin^2(2^n \theta)$.

STAGE 2 Proving it



Ask yourself Which proof method fits a statement about every term of a sequence defined by a rule from one term to the next?

Induction. The statement holds for $n = 0$, since $u_0 = \sin^2 \theta = \sin^2(2^0 \theta)$. Suppose $u_n = \sin^2(2^n \theta)$ for some n . Then

$$u_{n+1} = 4 \sin^2(2^n \theta) (1 - \sin^2(2^n \theta)) = (2 \sin(2^n \theta) \cos(2^n \theta))^2 = \sin^2(2^{n+1} \theta),$$

which is the statement for $n + 1$. So $u_n = \sin^2(2^n \theta)$ for all $n \geq 0$. ■

Notice that every starting value u in $[0, 1]$ can be written as $\sin^2 \theta$ (take $\theta = \arcsin \sqrt{u}$), so this solves the logistic map with $k = 4$ for every start in $[0, 1]$.

STAGE 3 Changing variables



Ask yourself Substitute $v_n = \alpha u_n + \beta$ into the rule for v_{n+1} . You need the result to be $u_{n+1} = 4u_n - 4u_n^2$. How many conditions does that give, and how many unknowns do you have?

Substituting,

$$\alpha u_{n+1} + \beta = -p(\alpha u_n + \beta)^2 + q(\alpha u_n + \beta) + r,$$

so

$$u_{n+1} = -p\alpha u_n^2 + (q - 2p\beta)u_n + \frac{-p\beta^2 + q\beta + r - \beta}{\alpha}.$$

For this to be $4u_n - 4u_n^2$ we need three things:

$$p\alpha = 4, \quad q - 2p\beta = 4, \quad -p\beta^2 + (q - 1)\beta + r = 0.$$

The first two fix $\alpha = 4/p$ and $\beta = (q - 4)/(2p)$. There are no unknowns left, so the third equation is a genuine condition on p, q and r . Substituting β and multiplying by $4p$,

$$-(q - 4)^2 + 2(q - 1)(q - 4) + 4pr = 0,$$

and expanding, $-(q^2 - 8q + 16) + (2q^2 - 10q + 8) + 4pr = 0$, that is $q^2 - 2q - 8 + 4pr = 0$, or

$$4pr = 8 + 2q - q^2. \quad \blacksquare$$



Pause and try Three equations but only two unknowns: this is why a condition appears. Can you see, before calculating, why a general quadratic map should not always be equivalent to (*)? (Think about what a linear change of variable can and cannot change.)

STAGE 4 The example

For $v_{n+1} = -v_n^2 + 2v_n + 2$ we have $p = 1, q = 2, r = 2$, and $4pr = 8 = 8 + 4 - 4$, so the condition holds. Then $\alpha = 4$ and $\beta = -1$, so $v_n = 4u_n - 1$, with u_n satisfying (*). Since $v_0 = 1, u_0 = \frac{1}{2} = \sin^2 \frac{\pi}{4}$, so

$$u_n = \sin^2\left(\frac{2^n \pi}{4}\right), \quad v_n = 4 \sin^2\left(\frac{2^n \pi}{4}\right) - 1 = 1 - 2 \cos\left(\frac{2^n \pi}{2}\right).$$

This gives $v_0 = 1, v_1 = 3$ and $v_n = -1$ for all $n \geq 2$. (Check directly: $-1 + 2 + 2 = 3$, then $-9 + 6 + 2 = -1$, then $-1 - 2 + 2 = -1$.) $\blacksquare$

The particular starting value was chosen so that the angle lands on a multiple of π and the sequence stops. That hides the interesting part, which comes next.

Looking back

- **Recognise the disguise.** The map $s \mapsto 4s(1 - s)$ became angle doubling once we wrote $s = \sin^2 \theta$.
- **Conjecture, then prove by induction.** Two terms suggested the pattern; induction proved it.
- **Count equations and unknowns.** A linear substitution has two free parameters, a quadratic has three coefficients, so one condition must be left over.

What's really going on

Why the sequence is unpredictable

The formula $u_n = \sin^2(2^n \theta)$ looks like the end of the story: we can compute any term directly. But look at what it says. Since $\sin^2$ has period π , only the value of $2^n \theta$ modulo π matters. Write $\theta = \pi x$ with $0 \leq x < 1$, and write x in binary:

$$x = 0.b_1 b_2 b_3 \dots \quad (\text{base } 2).$$

Multiplying by 2 shifts the binary point one place to the right, and taking the result modulo 1 throws away the digit that crosses the point. So u_n depends only on the binary digits of x from b_{n+1} onwards. Each step of the logistic map reads one more digit of the starting value and discards one from the front.

This is *sensitive dependence on initial conditions*, the defining feature of chaos. Two starting values that agree to 20 binary places (about six decimal places) produce sequences that agree for about 20 steps and then have nothing to do with each other (Figure 8.7.1). To predict u_{100} , you need to know u_0 to 100 binary places. Any error in measuring the starting population, however small, doubles at every step. The formula does not help: it simply makes the mechanism visible.

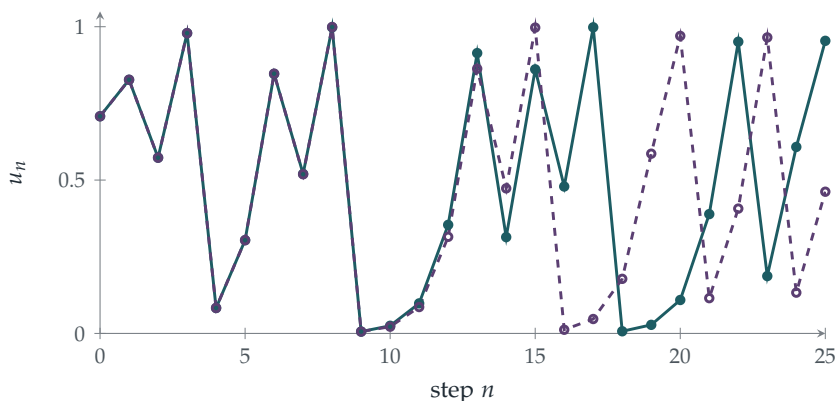


Figure 8.7.1 Two sequences of the logistic map with $k = 4$, starting from $\sin^2(1)$ (teal) and $\sin^2(1.00001)$ (plum, dashed). They are indistinguishable for about ten steps, visibly different by step 12, and unrelated after that: by step 12 the difference in angle, 10^{-5} , has been doubled twelve times, to about $2^{12} \times 10^{-5} \approx 0.04$.

Periodic points everywhere

The formula also finds every cycle of the map. The sequence returns to its starting value after N steps exactly when $\sin^2(2^N \theta) = \sin^2 \theta$, that is, when $2^N \theta \equiv \pm \theta \pmod{\pi}$. So $\theta = m\pi/(2^N - 1)$ or $\theta = m\pi/(2^N + 1)$ gives a point of period dividing N . For example $\theta = \pi/5$ satisfies $4\theta \equiv -\theta$, so $\sin^2 \frac{\pi}{5} \rightarrow \sin^2 \frac{2\pi}{5} \rightarrow \sin^2 \frac{4\pi}{5} = \sin^2 \frac{\pi}{5}$ is a 2-cycle. There are periodic points of every period, and they are dense in $[0, 1]$: in every

interval, however small, there is a starting value whose sequence repeats. Yet each one is unstable, because any error doubles. Dense periodic points that repel everything nearby are the second hallmark of chaos.

The road to chaos

For smaller k there is no formula, and the behaviour depends dramatically on k (Figure 8.7.2). For $1 < k < 3$ every sequence starting in $(0, 1)$ settles down to the fixed point $1 - 1/k$. At $k = 3$ the fixed point becomes unstable and a 2-cycle takes over (this is the subject of STEP II 2002 Q5; see the exercises). At $k = 1 + \sqrt{6} \approx 3.449$ the 2-cycle gives way to a 4-cycle, then an 8-cycle, with the doublings coming faster and faster, until at $k \approx 3.5699$ the cycles have infinite length and chaos begins. Inside the chaotic region are windows of order: the clearest, near $k = 3.83$, contains a 3-cycle, which is the subject of Section 8.8.

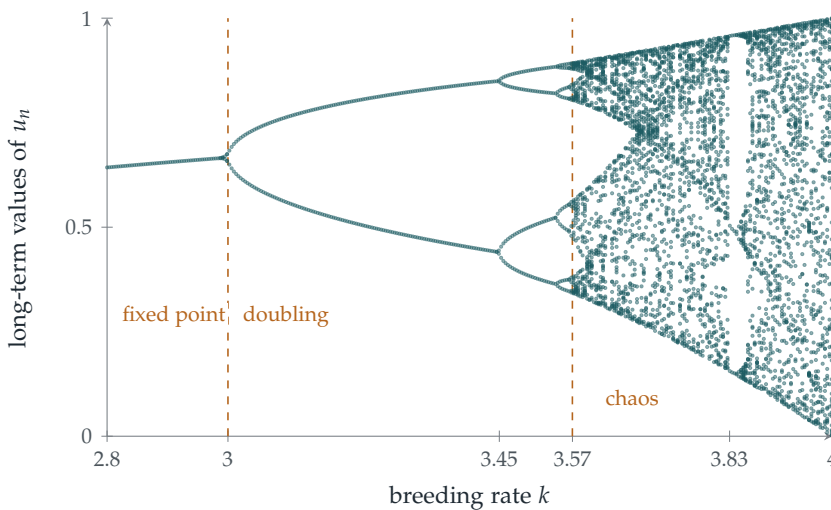


Figure 8.7.2 The bifurcation diagram of the logistic map. For each k , the map was iterated 300 times from $u_0 = 0.31416$ and the next 40 values plotted. A single fixed point splits into a 2-cycle at $k = 3$, then 4, 8, ..., until chaos sets in near $k = 3.57$. The pale vertical gap near $k = 3.83$ is the window containing a stable 3-cycle. At $k = 4$ the values fill all of $[0, 1]$.

Feigenbaum's constant

In 1975 Mitchell Feigenbaum, working at Los Alamos with a programmable pocket calculator, measured the values of k at which the period doubles: $k_1 = 3$, $k_2 \approx 3.4495$, $k_3 \approx 3.5441$, $k_4 \approx 3.5644$, ... He noticed that the gaps shrink by a constant factor,

$$\frac{k_n - k_{n-1}}{k_{n+1} - k_n} \rightarrow 4.6692 \dots,$$

and then, astonishingly, that the same number $4.6692\dots$ appears for $x \mapsto k \sin(\pi x)$ and for every other smooth map with a single hump. The constant is *universal*. It has since been measured in experiments on dripping taps, heated fluids and electronic circuits. Explaining why required ideas borrowed from the physics of phase transitions, and was only made rigorous by Oscar Lanford in 1982, with the help of a computer.

The same trick, three times

The substitution in part (ii) is an example of *conjugacy*: two maps are conjugate if a change of variable turns one into the other, and then they have exactly the same dynamics. The question shows that every quadratic map satisfying the condition is conjugate to the logistic map with $k = 4$, which in turn is conjugate (through $u = \sin^2 \theta$) to angle doubling. In Section 2.8 you met $x \mapsto x^2 - 2$, solved by $x = 2 \cos \theta$ with the angle doubling at each step; it satisfies the condition too (try it). So these are all the same map in disguise. In Section 8.8, a third member of the family appears: a map built from straight lines, the *tent map*.

The thinking skill

Change variables to see the structure

A good substitution can turn a complicated rule into a simple one. Look for a change of variable that turns the problem into one you already understand.

Here $u = \sin^2 \theta$ turned a quadratic map into angle doubling, and a linear substitution turned a whole family of quadratics into one. At university, conjugacy is a central tool: in dynamical systems, in linear algebra (diagonalising a matrix is choosing coordinates in which it is simple), and in group theory.

Try this next

1 Cycles of the logistic map. (From STEP II, 2002.) The sequence x_n satisfies $x_{n+1} = kx_n(1 - x_n)$.

- (a) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n .
- (b) Given that $x_0 = x_1 = x_2 = \dots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a .
- (c) Given instead that $x_0 = x_2 = x_4 = \dots = a$, with $a \neq 0, 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a . Hence show that a 2-cycle exists only when $k > 3$.

2 Squaring minus two. Show that $v_{n+1} = v_n^2 - 2$ satisfies the condition of part (ii) (write it as $-pv_n^2 + qv_n + r$ with $p = -1$). Find α and β , and check that your answer

agrees with $v_n = 2 \cos(2^n \phi)$.

- 3 A 3-cycle.** Use the formula $u_n = \sin^2(2^n \theta)$ to find a 3-cycle of $u_{n+1} = 4u_n(1 - u_n)$. (Hint: try $\theta = \pi/7$, or $\theta = \pi/9$.) How many points have period exactly 3?
- 4 Losing digits.** A computer stores numbers to about 53 binary places. If you iterate $u_{n+1} = 4u_n(1 - u_n)$ on a computer, roughly how many steps before the computed values bear no relation to the true sequence? Is the computed sequence then useless?

Further reading

- **Robert May**, “Simple mathematical models with very complicated dynamics”, *Nature* 261 (1976). A classic, still very readable.
- **James Gleick**, *Chaos: Making a New Science* (Penguin). The story of May, Feigenbaum and the discovery of chaos.
- **Steven Strogatz**, *Nonlinear Dynamics and Chaos* (CRC Press), Chapter 10. One-dimensional maps, the logistic map and Feigenbaum’s constant, at first-year undergraduate level.
- **Ian Stewart**, *Does God Play Dice?* (Penguin). A popular account of chaos, with a good chapter on the logistic map.

8.8 Period Three Implies Chaos

The question

Let $f(x) = 7 - 2|x|$. A sequence $u_0, u_1, u_2, \dots$ is defined by $u_0 = a$ and $u_n = f(u_{n-1})$ for $n > 0$.

- (i) (a) Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(x))$.
- (b) Find all solutions of the equation $f(f(x)) = x$.
- (c) Find the values of a for which the sequence $u_0, u_1, u_2, \dots$ has period 2.
- (d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \dots$ has period 2, but neither u_0 nor u_1 is equal to either of u_2 or u_3 .
- (ii) (a) Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(f(x)))$.
- (b) Consider the sequence $u_0, u_1, u_2, \dots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $f(f(f(x))) = x$.
- (c) Find a value of a such that the sequence $u_3, u_4, u_5, \dots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5 .

The hook

In 1975 Tien-Yien Li and James Yorke published a paper in the *American Mathematical Monthly* with one of the most famous titles in mathematics: “Period three implies chaos”. They proved that if a continuous function from an interval to itself has a point of period 3 (a starting value that returns after exactly three steps), then it has points of *every* period: 1, 2, 4, 5, 17, 1000, all of them. Their paper also introduced the word “chaos” into mathematics.

This question, set in 2025, is a hands-on tour of that theorem. The function $f(x) = 7 - 2|x|$ is built from two straight lines, so everything can be computed exactly. You find its 2-cycles and its 3-cycles, and in doing so see why the existence of one forces the existence of the rest.

Before you start



Ask yourself What does the graph of $y = 7 - 2|x|$ look like? Where does it cross the line $y = x$? Where does it send the interval $[-7, 7]$?

The graph is an upside-down V with its peak at $(0, 7)$ and gradients ± 2 . It meets $y = x$ where $x = 7 - 2x$ (with $x \geq 0$), giving $x = \frac{7}{3}$, and where $x = 7 + 2x$ (with $x < 0$), giving $x = -7$. These are the *fixed points*: start there and the sequence never moves. Also $f(-7) = f(7) = -7$ and $f(0) = 7$, so f maps $[-7, 7]$ onto itself, stretching it to twice its length and folding it in half.

Working it through

STAGE 1 The graph of $f \circ f$



Ask yourself As x goes from -7 to 7 , $f(x)$ goes from -7 up to 7 and back down to -7 . What does $f(f(x))$ do?

As x runs from -7 to 0 , $f(x)$ runs over the whole of $[-7, 7]$, so $f(f(x))$ traces out a complete copy of the V: up from -7 to 7 and down again. The same happens as x runs from 0 to 7 . So on $[-7, 7]$ the graph of $f \circ f$ is a W-shape turned upside down: four straight pieces of gradient ± 4 , each running between heights -7 and 7 , with peaks at $x = \pm \frac{7}{2}$ (where $f(x) = 0$) and a dip to -7 at $x = 0$ (Figure 8.8.1). Outside $[-7, 7]$ both graphs fall away steeply.

STAGE 2 Solving $f(f(x)) = x$



Ask yourself Each of the four straight pieces crosses the diagonal $y = x$ exactly once. Two of the crossings you already know. Why? How can you find the other two without expanding nested absolute values?

The fixed points $\frac{7}{3}$ and -7 certainly satisfy $f(f(x)) = x$. Any other solution p belongs to a 2-cycle $p \rightarrow q \rightarrow p$ with $q = f(p) \neq p$. Try the signs. If p and q are both positive, then $q = 7 - 2p$ and $p = 7 - 2q$, which forces $p = q = \frac{7}{3}$; if both are negative, the same argument gives -7 . So one is positive and one negative, say $p > 0 > q$:

$$q = 7 - 2p, \quad p = 7 + 2q.$$

Then $p = 7 + 14 - 4p$, so $p = \frac{21}{5}$ and $q = -\frac{7}{5}$, which indeed has $q < 0$.

Outside $[-7, 7]$ there are no solutions: for $x > 7$ we have $f(f(x)) = 21 - 4x < x$, and for $x < -7$ we have $f(f(x)) = 21 + 4x < x$. So the four solutions are

$$x = -7, \quad -\frac{7}{5}, \quad \frac{7}{3}, \quad \frac{21}{5}. \quad \blacksquare$$

The sequence has period 2 exactly when $a = \frac{21}{5}$ or $a = -\frac{7}{5}$. (At the fixed points the sequence is constant, with period 1.)

STAGE 3 Arriving at a cycle late

With $a = \frac{28}{5}$: $u_1 = 7 - \frac{56}{5} = -\frac{21}{5}$, $u_2 = 7 - \frac{42}{5} = -\frac{7}{5}$, $u_3 = 7 - \frac{14}{5} = \frac{21}{5}$, $u_4 = -\frac{7}{5}$, and so on. From u_2 onwards the sequence is the 2-cycle, but $u_0 = \frac{28}{5}$ and $u_1 = -\frac{21}{5}$ are not in it. ■



Ask yourself How could you have found $\frac{28}{5}$ yourself? Work backwards: which numbers does f send to $-\frac{7}{5}$?

$f(x) = -\frac{7}{5}$ means $|x| = \frac{21}{5}$, so $x = \pm\frac{21}{5}$. One of these is in the cycle; the other, $-\frac{21}{5}$, is a way in. Then $f(x) = -\frac{21}{5}$ means $|x| = \frac{28}{5}$. Because f is two-to-one, every point of $(-7, 7)$ has two preimages, and so every cycle has a tree of starting values that feed into it.

STAGE 4 Three-cycles



Pause and try Compute the sequences starting from $a = 1$ and $a = -\frac{7}{9}$.

From $a = 1$: $1 \rightarrow 5 \rightarrow -3 \rightarrow 1$. From $a = -\frac{7}{9}$: $-\frac{7}{9} \rightarrow \frac{49}{9} \rightarrow -\frac{35}{9} \rightarrow -\frac{7}{9}$. Both are 3-cycles.



Ask yourself How many solutions can $f(f(f(x))) = x$ have? Think about the shape of its graph, as in Stage 1.

Repeating the argument of Stage 1, each time we apply f again the number of straight pieces doubles. So on $[-7, 7]$ the graph of $f \circ f \circ f$ is a zigzag of eight pieces, of gradient ± 8 , each running the full height from -7 to 7 (Figure 8.8.2). Each piece crosses the diagonal exactly once, so there are exactly eight solutions in $[-7, 7]$, and (as before) none outside. We have found eight: the two fixed points and the six points of the two 3-cycles. So the solutions are

$$x = -7, \frac{7}{3}, 1, 5, -3, -\frac{7}{9}, \frac{49}{9}, -\frac{35}{9}. \quad \blacksquare$$

STAGE 5 A late arrival at a 3-cycle

Work backwards from the cycle $1 \rightarrow 5 \rightarrow -3$. We want u_3 in the cycle but u_2 not. The preimages of 1 satisfy $|x| = 3$, so $x = \pm 3$; -3 is in the cycle, so take $u_2 = 3$. The preimages of 3 satisfy $|x| = 2$; take $u_1 = 2$. The preimages of 2 satisfy $|x| = \frac{5}{2}$; take $u_0 = \frac{5}{2}$. Then

$$\frac{5}{2} \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 5 \rightarrow -3 \rightarrow 1 \rightarrow \dots,$$

and none of $\frac{5}{2}, 2, 3$ is in the cycle. So $a = \frac{5}{2}$ works. ■

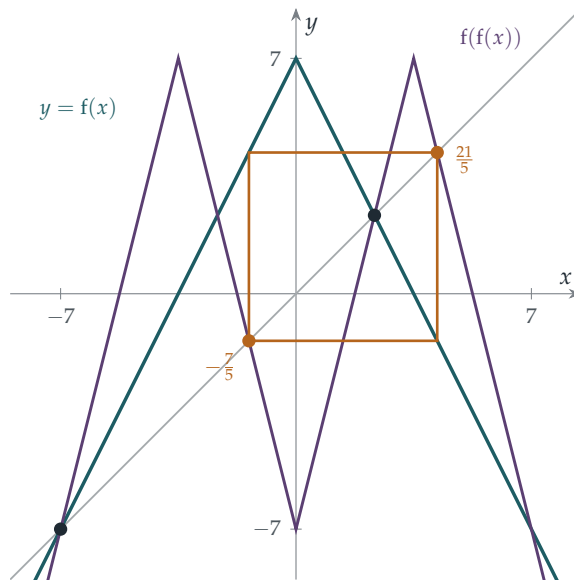


Figure 8.8.1 The graphs of $y = f(x)$ (teal), $y = f(f(x))$ (plum) and $y = x$ (grey). The plum graph meets the diagonal four times: at the fixed points -7 and $\frac{7}{3}$ (black), and at the 2-cycle $\frac{21}{5} \leftrightarrow -\frac{7}{5}$ (amber). The amber square is the *cobweb* of the 2-cycle: go up to the graph of f , across to the diagonal, and repeat.

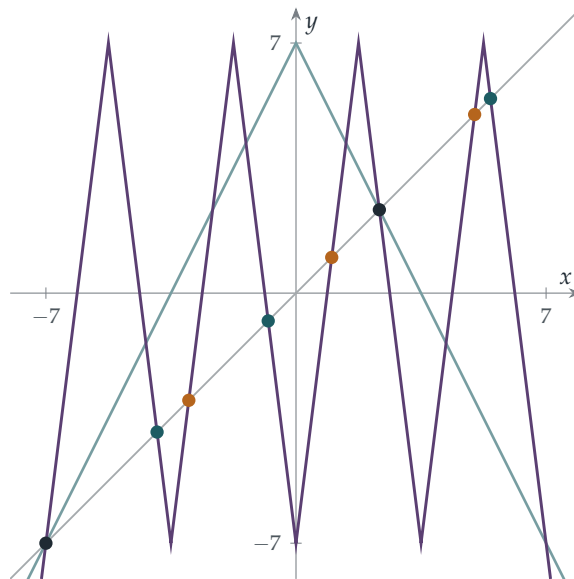


Figure 8.8.2 The graph of $y = f(f(f(x)))$ (plum) on $[-7, 7]$ is a zigzag of eight pieces with gradient ± 8 , each crossing the diagonal once. The eight crossings are the two fixed points (black), the 3-cycle $1 \rightarrow 5 \rightarrow -3$ (amber) and the 3-cycle $-\frac{7}{9} \rightarrow \frac{49}{9} \rightarrow -\frac{35}{9}$ (teal).

Looking back

- **Count by shape.** Instead of expanding nested absolute values, we counted the straight pieces of the graph. That told us how many solutions to look for, and so when we had found them all.
- **Use the structure of cycles.** Solutions of $f(f(x)) = x$ come as fixed points or in pairs; solutions of $f(f(f(x))) = x$ as fixed points or in threes.
- **Work backwards.** Preimages built the “late arrivals” in (i)(d) and (ii)(c).

What’s really going on

Cycles of every length

The counting in Stage 4 works for any number of steps. Write f^n for f applied n times. On $[-7, 7]$ the graph of f^n has 2^n straight pieces, each running the full height, so $f^n(x) = x$ has exactly 2^n solutions. Some of them have a shorter period d dividing n , but there are at most 2^d of those for each d , and

$$2^n > 2^{n/2} + 2^{n/3} + \cdots + 2^1 \quad (\text{summing over the divisors } d < n \text{ of } n)$$

because all the divisors less than n are at most $n/2$, and $2^1 + 2^2 + \cdots + 2^{\lfloor n/2 \rfloor} < 2^{\lfloor n/2 \rfloor + 1} \leq 2^n$. So for every n there are points whose period is exactly n . This function has cycles of every length.

Why period three forces everything

Li and Yorke’s theorem says this is not special to our function: *any* continuous g from an interval to itself with a 3-cycle has cycles of every length. The proof uses nothing more than the intermediate value theorem, and our 3-cycle shows how it goes.



Ask yourself Let $I_1 = [-3, 1]$ and $I_2 = [1, 5]$, the two intervals between the points of the cycle $1 \rightarrow 5 \rightarrow -3$. Where does f send each of them?

On I_2 , f is decreasing from $f(1) = 5$ to $f(5) = -3$, so $f(I_2) = [-3, 5] = I_1 \cup I_2$. On I_1 , f goes from $f(-3) = 1$ up to $f(0) = 7$ and down to $f(1) = 5$, so $f(I_1) = [1, 7]$, which contains I_2 . Notice that only the three points of the cycle were used: any continuous function with a 3-cycle $a \rightarrow b \rightarrow c$, $a < b < c$, covers its intervals in the same way. We can summarise it as a diagram of which interval covers which (Figure 8.8.3):

$$I_1 \rightarrow I_2, \quad I_2 \rightarrow I_1, \quad I_2 \rightarrow I_2.$$

The theorem then follows from two facts, both consequences of the intermediate value theorem.

- If $g(I) \supseteq I$ then g has a fixed point in I . (Apply the intermediate value theorem to $g(x) - x$.)

► If $g(I) \supseteq J$ then there is a smaller closed interval $I' \subseteq I$ with $g(I') = J$ exactly.

To find a point of period n , follow the loop $I_2 \rightarrow I_2 \rightarrow \cdots \rightarrow I_2 \rightarrow I_1 \rightarrow I_2$ in the diagram: $n - 1$ steps inside I_2 and one step out to I_1 and back. Using the second fact repeatedly, we can shrink I_2 to an interval of starting points whose orbits follow this itinerary, and g^n maps that interval onto all of I_2 . The first fact gives a point x with $g^n(x) = x$ that visits I_1 only at step $n - 1$. Its period cannot be smaller than n , because then the visit to I_1 would be repeated earlier. (There is one subtlety, since I_1 and I_2 share the point 1; checking that it causes no trouble is an exercise.)



Figure 8.8.3 The covering diagram of a 3-cycle. An arrow from one interval to another means the image of the first contains the second. Every loop in the diagram gives a cycle of the function: the loop $I_2 \rightarrow I_2$ gives a fixed point, $I_1 \rightarrow I_2 \rightarrow I_1$ a 2-cycle, and going round the loop at I_2 $n - 1$ times before visiting I_1 gives a cycle of length n .

Sharkovskii's theorem

Li and Yorke were not the first. In 1964 the Ukrainian mathematician Oleksandr Sharkovskii had proved a far more complete result, published in Russian in the *Ukrainian Mathematical Journal* and unknown in the West for a decade. Order the positive integers as

$$3, 5, 7, 9, \dots, 2 \cdot 3, 2 \cdot 5, \dots, 2^2 \cdot 3, 2^2 \cdot 5, \dots, \dots, 2^3, 2^2, 2, 1 :$$

first the odd numbers from 3 upwards, then twice the odd numbers, then four times the odd numbers, and so on, ending with the powers of 2 in decreasing order. Sharkovskii's theorem says that if a continuous function on an interval has a cycle of length m , then it has cycles of every length that comes after m in this list. Since 3 comes first, period 3 implies every period. And the theorem is sharp: for every m there is a function with a cycle of length m and no cycles of lengths earlier in the list. The powers of 2 at the end are exactly the period-doubling cascade of the logistic map in Section 8.7, and the 3-cycle appears there at $k = 1 + \sqrt{8}$.

The tent map and the logistic map

Substitute $x = 14y - 7$, which maps $[0, 1]$ onto $[-7, 7]$. A short calculation shows that f becomes the *tent map*

$$T(y) = \begin{cases} 2y & 0 \leq y \leq \frac{1}{2}, \\ 2 - 2y & \frac{1}{2} \leq y \leq 1, \end{cases}$$

and the further substitution $u = \sin^2(\pi y/2)$ turns the tent map into the logistic map $u \mapsto 4u(1-u)$. So the function in this question is the chaotic logistic map of Section 8.7, straightened out. Its eight solutions of $f^3(x) = x$ correspond to the eight solutions of $\sin^2(8\theta) = \sin^2 \theta$ in $[0, \frac{\pi}{2}]$.

The thinking skill

Existence by the intermediate value theorem

To show that a solution exists, you do not always need to find it. Show that a continuous function changes sign, or that an interval is mapped over itself.

In this question every solution could be computed. But Li and Yorke's theorem applies to functions we cannot compute with at all, and it proves the existence of infinitely many cycles using only the intermediate value theorem and a diagram of which interval covers which. At university this style of argument, existence without construction, appears in analysis (the intermediate value theorem itself), in topology (fixed point theorems) and throughout dynamical systems.

Try this next

- 1 A 4-cycle.** Use the covering diagram to predict where a 4-cycle of f should be (three steps in I_2 , one in I_1). Then find it exactly. (*Hint:* the cycle has three positive points and one negative point; write down the four linear equations.)
- 2 The subtlety.** In the proof that period 3 implies period n , explain why the point x found cannot be the point 1 (or any point of the original 3-cycle) when $n \geq 4$.
- 3 Period 5 without period 3.** Sketch the graph of a continuous function on $[1, 5]$ with $g(1) = 3$, $g(3) = 4$, $g(4) = 2$, $g(2) = 5$, $g(5) = 1$, joining these points with straight lines. Draw its covering diagram for the intervals $[1, 2]$, $[2, 3]$, $[3, 4]$, $[4, 5]$. Can you find a loop of length 3? (Sharkovskii's theorem says there is no 3-cycle.)
- 4 The tent map.** Check that $x = 14y - 7$ turns f into the tent map. Then show that the tent map doubles y written in binary, flipping all the digits whenever the leading digit is 1, and use this to find its 3-cycles.

Further reading

- **Tien-Yien Li & James Yorke**, "Period three implies chaos", *American Mathematical Monthly* 82 (1975). Short, and readable with the intermediate value theorem.
- **Robert Devaney**, *A First Course in Chaotic Dynamical Systems* (CRC Press). Written for undergraduates; covers the tent map, the logistic map and Sharkovskii's theorem

with full proofs.

- **Keith Burns & Boris Hasselblatt**, “The Sharkovsky theorem: a natural direct proof”, *American Mathematical Monthly* 118 (2011). A modern proof using covering diagrams.

What We Have Learned

We began by saying that most differential equations cannot be solved, and that the art is to understand them anyway. We asked three questions: does the present determine the future, what can we know without solving, and what happens in the long run? Here is what the chapter's eight questions showed.

1. Does the present determine the future?

Usually, but not always. In Section 8.1, the equation $(y')^2 = 4y$ had infinitely many solutions through every point of the x -axis: a solution could rest at $y = 0$ for as long as it liked and then leave. The culprit was the square root, whose infinite slope at zero breaks the Lipschitz condition of the Picard–Lindelöf theorem. The same mechanism explains why a leaking bucket cannot tell you when it emptied, and it lies behind Norton's dome, a puzzle about determinism in classical mechanics.

Even when the future is determined, it may not be predictable. The logistic map of Section 8.7 has an exact formula, $u_n = \sin^2(2^n \theta)$, and yet every step doubles any error in the starting value. Determinism in principle is not predictability in practice.

2. What can we know without solving?

A great deal.

- ▶ **Energy.** In Section 8.2, multiplying the equation by y' produced a quantity that could not increase, and that bounded the solution of an equation nobody can solve. Such quantities are called *Lyapunov functions*, and they are the main tool for proving stability.
- ▶ **Equilibria and stability.** In Section 8.3, the sign of the growth rate predicted the fate of a fish population, and showed that harvesting beyond a threshold causes sudden collapse.
- ▶ **The right coordinates.** In Section 8.4, polar coordinates turned a coupled system into a single equation for r , showing every solution spiralling onto a circle. In Section 8.5, symmetry reduced four chasing dogs to one, and dividing to eliminate time gave the shape of their paths.
- ▶ **Structure from an identity.** In Section 8.6, the Wronskian showed that the solutions of an equation are orthogonal, without integrating a single one of them.
- ▶ **Counting and covering.** In Section 8.8, counting the pieces of a graph found every cycle, and the intermediate value theorem proved the existence of cycles of every length.

3. What happens in the long run?

The chapter met every kind of long-term behaviour. Solutions settled to stable equilibria (Section 8.3), approached a repeating cycle (Section 8.4), spiralled into a point in finite time (Section 8.5), or behaved chaotically (Section 8.7). For the logistic map, a single parameter moves the system through all of these: a fixed point, a cascade of period doublings governed by Feigenbaum’s universal constant, and finally chaos, with windows of order such as the 3-cycle. And Section 8.8 showed that once a 3-cycle appears, cycles of every length must exist, the first step in Sharkovskii’s ordering of the integers.

4. A toolkit for differential equations and dynamics

Method	Where you used it
Looking for singular solutions	Non-uniqueness (8.1)
A quantity that cannot increase	Bounding (8.2)
Sign of the growth rate	Harvesting (8.3)
Coordinates that match the symmetry	Limit cycle (8.4); pursuit (8.5)
Substitution to linearise	Bernoulli equation (8.4)
The Wronskian	Orthogonality (8.6)
Conjugacy	Logistic map (8.7)
Induction	$u_n = \sin^2(2^n \theta)$ (8.7)
Counting pieces of a graph	Cycles of f^n (8.8)
Intermediate value theorem	Period three (8.8)

Where next

- Energy and conservation laws return in mechanics, starting with Section 10.1.
- Linear differential equations with several unknowns, solved by eigenvectors, govern vibrating systems in Section 11.2; the nonlinear pendulum, which cannot be solved in elementary functions, is in Section 11.3.
- Iterating a map to find its fixed points decides whether a family name dies out in Section 12.15, and long-run behaviour of random sequences is the subject of Section 12.16.

If you want to go further before university

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*:** the best first book on dynamical systems, full of applications and pictures. Strogatz’s recorded lectures based on it are free online.
- **James Gleick, *Chaos: Making a New Science*:** the story of how chaos was discovered.

- ▶ **George Simmons, *Differential Equations with Applications and Historical Notes*:** a classic first course, rich in history.
- ▶ **Robert Devaney, *A First Course in Chaotic Dynamical Systems*:** iterated maps, with proofs, at first-year level.

Laplace imagined that, knowing the present exactly, one could compute the whole future. This chapter has shown where that dream stops. Uniqueness can fail; and where it holds, chaos can amplify the smallest uncertainty until prediction is impossible. But it has also shown that a great deal can be understood without a formula: bounds, equilibria, cycles and their stability, and the structure of chaos itself. Moving from “solve the equation” to “understand the solutions” is one of the most important shifts in thinking that a university course in mathematics asks of you.