

CHAPTER

7

Linear Algebra and Geometry

A matrix is not a table of numbers. It is a transformation of space, and it has a life of its own.

IN THIS CHAPTER

- | | | |
|-----|-------------------------------------|---------------|
| 7.1 | Rules That Fail for Matrices | STEP II 1993 |
| 7.2 | Stretching Without Turning | STEP III 1990 |
| 7.3 | What Makes a Determinant | STEP II 2019 |
| 7.4 | Matrices That Return to the Start | STEP II 2020 |
| 7.5 | Rotating in Three Dimensions | STEP III 2022 |
| 7.6 | Hyperbolic Rotations and Relativity | STEP III 1991 |
| 7.7 | Quantities That Never Change | STEP III 2024 |
| 7.8 | Four Squares | STEP III 1994 |
| 7.9 | Only Five Platonic Solids | STEP III 1999 |

Transformations of Space

We think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.

— Irving Kaplansky, on his and Paul Halmos's approach to linear algebra

At school, a matrix usually appears as a tool for solving simultaneous equations, or as a recipe for transforming points: multiply the matrix by a column vector and read off the new coordinates. Both are useful, but they hide what makes matrices one of the most important ideas in mathematics. A matrix is a *linear transformation*: a way of stretching, rotating, reflecting and shearing space, and matrices can be added, multiplied, raised to powers and differentiated, just like numbers. Except that they are not like numbers. Their multiplication depends on the order, they can multiply to zero, and they have hidden invariants (trace, determinant, eigenvalues) that capture their essential behaviour regardless of how they are written down.

Why linear algebra matters

Linear algebra is probably the most widely used branch of mathematics. Every computer graphics engine, from video games to film animation, is a machine for multiplying matrices: every rotation of a character, every camera angle, every perspective projection is a matrix. The search engines and recommendation systems of the internet are built on eigenvectors. Engineers find the vibration frequencies of bridges and buildings from eigenvalues; statisticians find the main patterns in data with them; quantum mechanics is written entirely in the language of linear algebra. And modern machine learning, at its core, is very large amounts of matrix multiplication.

There is a reason for this ubiquity. Linear problems are the ones we can solve completely, and almost every non-linear problem looks linear if you zoom in close enough. That is the idea of calculus, and it means that linear algebra is the language in which the local behaviour of almost everything is described.

What you will meet at university

Linear algebra is one of the two pillars of the first year, alongside analysis. You will meet *vector spaces* (sets where you can add and scale, of which the plane and three-dimensional space are just the first examples) and *linear maps* between them, and learn that matrices are simply linear maps written in coordinates. You will study determinants, eigenvalues and eigenvectors, and the *spectral theorem*, which says that symmetric matrices can be diagonalised by a rotation. Later courses use this everywhere: in differential equations (Chapter 8), in geometry (where rotations and reflections form

groups), in the theory of *Lie groups*, which describe continuous symmetries, and in functional analysis, which is linear algebra in infinitely many dimensions.

The questions that run through this chapter

Three big questions

- 1. **How are matrices like numbers, and how are they different?** Which rules of arithmetic survive, and which fail?
- 2. **What does a matrix really do?** How can we see the geometry of a transformation, and which numbers capture it independently of coordinates?
- 3. **What stays the same?** Which quantities are preserved by transformations, or by motion, and what do they tell us?

The chapter starts with the rules of matrix algebra (Section 7.1), then turns to the geometry of transformations: eigenvectors, determinants, matrices of finite order, rotations in three dimensions and the hyperbolic rotations of special relativity. It continues with conserved quantities of evolving matrices, the four-square identity from determinants, and ends with a counting argument, Euler’s formula, that classifies the Platonic solids.

Ideas in this chapter

§	Topic	Idea
7.1	Rules that fail	Non-commutativity; zero divisors
7.2	Eigenvectors	Directions a transformation does not rotate
7.3	The determinant	Characterised by multiplicativity; signed area
7.4	Matrices of finite order	Trace, determinant and Cayley–Hamilton
7.5	Rotations in 3D	Rodrigues’ formula from the cross product
7.6	Lorentz boosts	Hyperbolic angles add
7.7	Conserved quantities	The trace of a commutator is zero
7.8	Four squares	Multiplicative determinants
7.9	Platonic solids	Double counting and Euler’s formula

How to read this chapter

For every matrix in this chapter, draw what it does. Take the unit square, or the unit circle, or a few vectors, and sketch their images. A two-minute sketch will often show you the answer (an eigenvector, a counterexample, an invariant line) before you start calculating, and it will help you check the calculation afterwards.

7.1 Rules That Fail for Matrices

The question

In this question, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ are non-zero 2×2 real matrices.

Are the following assertions true or false? You must provide a proof or a counterexample in each case.

- (i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.
- (ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.
- (iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.
- (iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

The hook

Matrices can be added and multiplied, so it is tempting to treat them like numbers. This question tests that temptation with four “rules” that are true for numbers, and asks which survive for matrices. Most of them fail, and the way they fail is instructive. Matrix multiplication is not commutative; there are non-zero matrices whose product is zero; and a quadratic equation for matrices can have infinitely many solutions.

These failures are not defects. They are the reason matrices are useful: they describe transformations of space, and doing two transformations in different orders genuinely gives different results. In 1925 Werner Heisenberg discovered that the quantities of quantum mechanics behave like matrices rather than numbers, and that the failure of $\mathbf{AB} = \mathbf{BA}$ is the mathematical form of the uncertainty principle.

Before you start



Ask yourself For real numbers, why is it true that $ab = 0$ implies $a = 0$ or $b = 0$? Which property of numbers does the argument use, and does it hold for matrices?

If $ab = 0$ and $a \neq 0$, divide by a to get $b = 0$. The argument needs every non-zero number to have an inverse. Non-zero matrices need not have inverses (those with determinant zero do not), so the argument breaks down. Keep the *geometric* picture in mind too: a matrix is a transformation of the plane, and a product $\mathbf{AB}$ means “do $\mathbf{B}$, then $\mathbf{A}$ ”.

Working it through

STAGE 1 Can $\mathbf{AB} = \mathbf{0}$ but $\mathbf{BA} \neq \mathbf{0}$?

Ask yourself Think geometrically. You want $\mathbf{B}$ to send everything into a line that $\mathbf{A}$ then squashes to zero. But in the other order, $\mathbf{A}$ should not squash everything that $\mathbf{B}$ then kills.

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which projects onto the x -axis (killing the y -axis), and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which sends (x, y) to $(0, x)$, into the y -axis. Then

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \mathbf{0}.$$

The assertion is **false** (Figure 7.1.1). ■

STAGE 2 The difference of two squares

Expanding carefully, keeping the order of every product:

$$(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} - \mathbf{BA} - \mathbf{B}^2.$$

This equals $\mathbf{A}^2 - \mathbf{B}^2$ exactly when $\mathbf{AB} = \mathbf{BA}$. With the matrices of Stage 1, $\mathbf{AB} - \mathbf{BA} = -\mathbf{BA} \neq \mathbf{0}$, so the assertion is **false**. ■

STAGE 3 Non-zero solutions of $\mathbf{AX} = \mathbf{0}$

Ask yourself If $\det \mathbf{A} \neq 0$, what can you do to $\mathbf{AX} = \mathbf{0}$? If $\det \mathbf{A} = 0$, can you find a non-zero column vector $\mathbf{v}$ with $\mathbf{Av} = \mathbf{0}$?

If $\det \mathbf{A} \neq 0$, then $\mathbf{A}^{-1}$ exists and $\mathbf{X} = \mathbf{A}^{-1}\mathbf{AX} = \mathbf{0}$: no non-zero solution.

If $\det \mathbf{A} = 0$, write $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad = bc$. The vectors $\begin{pmatrix} b \\ -a \end{pmatrix}$ and $\begin{pmatrix} d \\ -c \end{pmatrix}$ satisfy $\mathbf{A} \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} ab - ba \\ cb - da \end{pmatrix} = \mathbf{0}$, and similarly for the other. Since $\mathbf{A} \neq \mathbf{0}$, at least one of them is non-zero; call it $\mathbf{v}$. Then the matrix $\mathbf{X}$ whose two columns are both $\mathbf{v}$ is non-zero and $\mathbf{AX} = \mathbf{0}$. So the assertion is **true**. ■

STAGE 4 How many square roots?

Pause and try Find non-zero $\mathbf{A}$ and $\mathbf{B}$ for which $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$ has infinitely many solutions. (*Hint*: make the left-hand side a perfect square, $(\mathbf{X} - \mathbf{I})^2$, and think about matrices whose square is zero.)

Take $\mathbf{A} = -2\mathbf{I}$ and $\mathbf{B} = \mathbf{I}$, both non-zero. The equation becomes $(\mathbf{X} - \mathbf{I})^2 = \mathbf{0}$. Any matrix $\mathbf{N}$ with $\mathbf{N}^2 = \mathbf{0}$ gives a solution $\mathbf{X} = \mathbf{I} + \mathbf{N}$, and there are infinitely many such $\mathbf{N}$:

$$\mathbf{N} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \mathbf{N}^2 = \mathbf{0} \text{ for every real } t.$$

So $\mathbf{X} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is a solution for every t , and the assertion is **false**. ■

Looking back

- **Order matters.** Every expansion must keep products in order; $\mathbf{AB}$ and $\mathbf{BA}$ are usually different.
- **Zero divisors.** Non-zero matrices can multiply to zero, which breaks the argument behind “a quadratic has at most two roots”.
- **Think of transformations.** Each counterexample was easy to find by picturing what the matrices do to the plane.

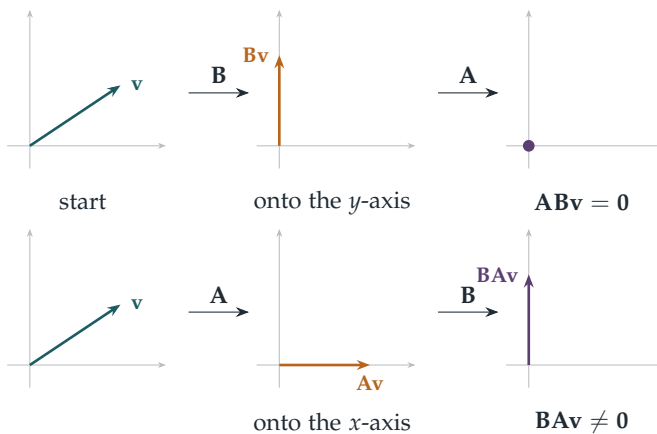


Figure 7.1.1 Why $\mathbf{AB} = \mathbf{0}$ but $\mathbf{BA} \neq \mathbf{0}$. Top: $\mathbf{B}$ sends every vector onto the y -axis, which $\mathbf{A}$ (projection onto the x -axis) then crushes to zero. Bottom: in the other order, $\mathbf{A}$ keeps the x -component, and $\mathbf{B}$ moves it onto the y -axis, where it survives.

What's really going on

Matrices as an algebra

Arthur Cayley introduced matrix algebra in 1858, treating matrices as single objects that can be added and multiplied, rather than just as tables of coefficients. He noticed at once that multiplication is not commutative, and he proved the *Cayley–Hamilton*

theorem: every square matrix satisfies its own characteristic equation. For a 2×2 matrix with trace t and determinant d , this says $\mathbf{M}^2 - t\mathbf{M} + d\mathbf{I} = \mathbf{0}$, a fact we use repeatedly in this chapter (Section 7.4). The 2×2 real matrices form a *ring*: you can add, subtract and multiply, but multiplication is not commutative, and there are *zero divisors*, non-zero elements whose product is zero.

Infinitely many square roots

Stage 4 can be made even more striking. The equation $\mathbf{X}^2 = \mathbf{I}$ has the obvious solutions $\pm \mathbf{I}$, but also every reflection of the plane in a line through the origin, $\mathbf{X} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$, since reflecting twice does nothing. That is a whole circle of square roots of $\mathbf{I}$. A polynomial of degree n with number coefficients has at most n roots because numbers have no zero divisors; matrices do, and the count fails.

Non-commutativity in physics

In classical mechanics, position x and momentum p are numbers and $xp = px$. In quantum mechanics they become operators (infinite matrices) with $xp - px = i\hbar$, where $\hbar$ is Planck's constant divided by 2π . This single non-commutation relation implies Heisenberg's uncertainty principle: position and momentum cannot both be known exactly. When Heisenberg discovered the rules of quantum mechanics in 1925 he did not know what matrices were; Max Born recognised that the strange multiplication rule Heisenberg had invented was matrix multiplication.

The thinking skill

Test familiar rules in a new setting

When you meet a new kind of object, check which familiar rules still hold, and find out exactly which property each rule depends on.

The rule " $ab = 0$ implies $a = 0$ or $b = 0$ " depends on division; " $(a - b)(a + b) = a^2 - b^2$ " depends on commutativity. Knowing what each rule rests on tells you when you can still use it.

Try this next

1 When do they commute? Find all 2×2 matrices that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2 Square roots of zero. Find all 2×2 real matrices $\mathbf{N}$ with $\mathbf{N}^2 = \mathbf{0}$. (*Hint:* use the Cayley–Hamilton theorem to show $\text{tr } \mathbf{N} = \det \mathbf{N} = 0$.)

- 3 Reflections.** Show that every reflection matrix $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$ squares to $\mathbf{I}$, and find the line it reflects in.
- 4 Cayley–Hamilton.** Prove that every 2×2 matrix satisfies $\mathbf{M}^2 - (\text{tr } \mathbf{M})\mathbf{M} + (\det \mathbf{M})\mathbf{I} = \mathbf{0}$ by direct calculation.

Further reading

- **Gilbert Strang**, *Introduction to Linear Algebra* (Wellesley-Cambridge). The most popular first linear algebra text, with an emphasis on understanding.
- **Sheldon Axler**, *Linear Algebra Done Right* (Springer; free online). A more abstract approach, centred on linear maps rather than matrices.
- **Jim Baggott**, *The Quantum Story* (Oxford University Press). The history of quantum mechanics, including Heisenberg’s matrix mechanics.

7.2 Stretching Without Turning

The question

The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$ is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show that T leaves the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ unchanged in direction but multiplied by a scalar, and that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is similarly transformed.

The circle C whose equation is $x^2 + y^2 = 1$ transforms under T to a curve E . Show that E has equation

$$8X^2 + 12XY + 17Y^2 = 80,$$

and state the area of the region bounded by E . Show also that the greatest value of X on E is $2\sqrt{17/5}$.

Find the equation of the tangent to E at the point which corresponds to the point $\frac{1}{5}(3, 4)$ on C .

The hook

A matrix transformation usually moves every vector to a new direction. But almost every matrix has a few special directions it does not turn at all, only stretching or shrinking along them. These are its *eigenvectors* (from the German *eigen*, “own” or “characteristic”), and the stretch factors are its *eigenvalues*. Once you know them, the transformation becomes transparent: in the right coordinates it is just a stretch along two axes.

This question finds the eigenvectors of a particular matrix and uses them to understand what it does to a circle. Eigenvectors are one of the most useful ideas in all of applied mathematics. They describe the natural vibrations of a bridge, the principal axes of a spinning object, the energy levels of an atom, the most important directions in a cloud of data, and the ranking of web pages by Google’s original PageRank algorithm.

Before you start



Ask yourself What does it mean for T to leave a vector “unchanged in direction but multiplied by a scalar”? Write it as an equation.

It means $T\mathbf{v} = \lambda\mathbf{v}$ for some number λ : $\mathbf{v}$ is an eigenvector with eigenvalue λ . Notice that the two vectors in the question, $(1, 2)$ and $(2, -1)$, are perpendicular. That is not a coincidence: the matrix is *symmetric*, and symmetric matrices always have perpendicular eigenvectors.

Working it through

STAGE 1 The eigenvectors

$$T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 9-4 \\ -2+12 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 20 \\ -10 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

So $(1, 2)$ is stretched by 2 and $(2, -1)$ by 4. ■

STAGE 2 The image of the circle



Ask yourself To find the equation of E , express x and y in terms of X and Y (invert the matrix), and substitute into $x^2 + y^2 = 1$.

The matrix $\frac{2}{5} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ has determinant $\frac{4}{25}(54 - 4) = 8$, and its inverse is $\frac{1}{20} \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$. So $x = \frac{6X+2Y}{20}$ and $y = \frac{2X+9Y}{20}$, and $x^2 + y^2 = 1$ becomes

$$(6X + 2Y)^2 + (2X + 9Y)^2 = 400 \implies 40X^2 + 60XY + 85Y^2 = 400,$$

which, divided by 5, is $8X^2 + 12XY + 17Y^2 = 80$. ■



Ask yourself What shape is E , and what is its area? Use the eigenvectors.

The unit circle has radius 1 in every direction. In the direction of $(2, -1)$ it is stretched by 4, and in the perpendicular direction $(1, 2)$ by 2. So E is an ellipse with semi-axes 4 and 2, along the eigenvector directions (Figure 7.2.1), and its area is $\pi \times 4 \times 2 = 8\pi$. (Equivalently, any linear map multiplies areas by $|\det T| = 8$, and the circle has area π .)

STAGE 3 The greatest value of X

Ask yourself On C , $X = \frac{2}{5}(9x - 2y)$. How large can $9x - 2y$ be when $x^2 + y^2 = 1$?

$9x - 2y$ is the dot product of $(9, -2)$ with the unit vector (x, y) , so its largest value is $|(9, -2)| = \sqrt{85}$, attained when (x, y) points along $(9, -2)$. Hence the greatest value of X is $\frac{2}{5}\sqrt{85} = 2\sqrt{\frac{85}{25}} = 2\sqrt{\frac{17}{5}}$. ■

STAGE 4 A tangent

Pause and try The point $\frac{1}{5}(3, 4)$ on C has tangent line $3x + 4y = 5$. A linear map sends tangent lines to tangent lines. Use this to find the tangent to E .

Substituting $x = \frac{6X+2Y}{20}$, $y = \frac{2X+9Y}{20}$ into $3x + 4y = 5$:

$$\frac{3(6X+2Y) + 4(2X+9Y)}{20} = 5 \implies 26X + 42Y = 100,$$

that is, $13X + 21Y = 50$. ■

(Check: the point itself is $T(\frac{3}{5}, \frac{4}{5}) = (\frac{38}{25}, \frac{36}{25})$, and $13 \cdot \frac{38}{25} + 21 \cdot \frac{36}{25} = \frac{494+756}{25} = 50$.)

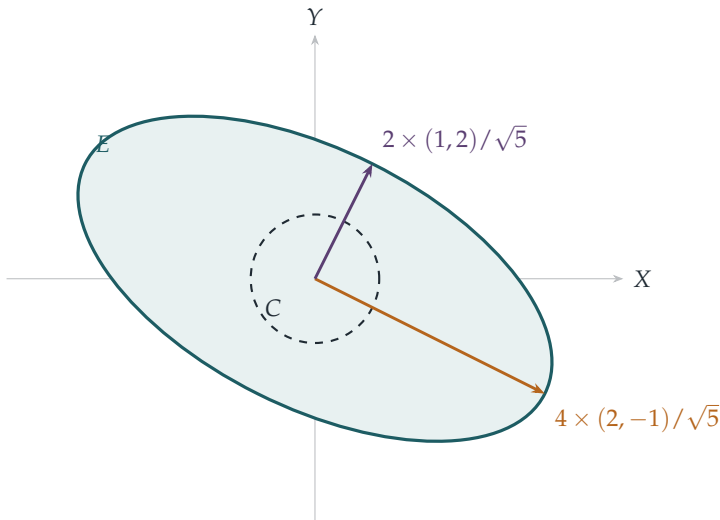


Figure 7.2.1 The unit circle C (dashed) and its image E under T . Along the eigenvector $(2, -1)$ the circle is stretched by a factor of 4; along the perpendicular eigenvector $(1, 2)$, by a factor of 2. So E is an ellipse with semi-axes 4 and 2 along those directions, and area 8π .

Looking back

► **Eigenvectors are the natural axes.** In the coordinates given by the eigenvectors, T is

just a stretch by 4 and by 2, and its effect on a circle is obvious.

- **Determinants measure area.** The area of E is $|\det T|$ times the area of C .
- **Linear maps preserve tangency.** Transforming the circle's tangent gave the ellipse's tangent with almost no calculation.

What's really going on

The spectral theorem

Every real symmetric matrix has a full set of perpendicular eigenvectors, with real eigenvalues. This is the *spectral theorem*, one of the central results of a first linear algebra course. It says that every symmetric linear map is, in suitable perpendicular coordinates, just a stretch along each axis. Equivalently, every quadratic form like $8X^2 + 12XY + 17Y^2$ can be rotated into the form $\lambda_1 u^2 + \lambda_2 v^2$, which is why every central conic is an ellipse or hyperbola with perpendicular axes (the *principal axis theorem*). The idea goes back to Cauchy in the 1820s, who used it to find the principal axes of quadric surfaces and of the stress in a solid body.

Eigenvectors everywhere

Here are a few of the places eigenvectors appear.

- **Vibrations.** A system of masses and springs vibrates in *normal modes*, patterns in which every part moves in step at a single frequency. These are eigenvectors of a matrix, and the frequencies come from the eigenvalues (Section 11.2).
- **Data.** The directions in which a cloud of data points is most spread out are the eigenvectors of its covariance matrix. Finding them is *principal component analysis*, one of the basic tools of statistics and machine learning (Section 13.7).
- **Quantum mechanics.** The possible results of measuring a quantity are the eigenvalues of a matrix (or operator), and the energy levels of an atom are eigenvalues.
- **Web search.** Google's PageRank, the original idea behind the search engine, ranks pages by the entries of the eigenvector, with eigenvalue 1, of a huge matrix describing the links between pages.

The thinking skill

Choose coordinates adapted to the transformation

To understand a linear transformation, find the directions it does not rotate. In those coordinates it becomes simple.

This is the same idea as choosing the Fermat point as origin in Section 5.6, or changing variables to simplify a map in Section 2.8: the right coordinates make the structure obvious.

Try this next

- 1 Finding eigenvalues.** Show that the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the roots of $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$, and check that the matrix in the question gives $\lambda = 2$ and 4 .
- 2 Perpendicular eigenvectors.** Show that if $\mathbf{M}$ is symmetric and $\mathbf{M}\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{M}\mathbf{v} = \mu\mathbf{v}$ with $\lambda \neq \mu$, then $\mathbf{u} \cdot \mathbf{v} = 0$. (*Hint:* compute $\mathbf{v} \cdot \mathbf{M}\mathbf{u}$ in two ways.)
- 3 A rotation.** Show that the rotation matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ has no real eigenvectors unless θ is a multiple of π . What are its complex eigenvalues?
- 4 Powers.** Using the eigenvectors, find $T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (*Hint:* write $(1, 0)$ as a combination of $(1, 2)$ and $(2, -1)$.)

Further reading

- **Gilbert Strang**, *Introduction to Linear Algebra* (Wellesley-Cambridge). Chapter 6 is an excellent introduction to eigenvalues and their applications.
- **Kurt Bryan & Tanya Leise**, “The \$25,000,000,000 eigenvector: the linear algebra behind Google”, *SIAM Review* 48 (2006). Readable and freely available.
- **3Blue1Brown**, *Essence of Linear Algebra* (video series, free online). Beautiful animations of exactly the geometric ideas in this section.

7.3 What Makes a Determinant

The question

The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $f(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that $f(\mathbf{MN}) = f(\mathbf{M})f(\mathbf{N})$ for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

- (i) You are given that there is a matrix $\mathbf{M}$ such that $f(\mathbf{M}) \neq 0$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $f(\mathbf{MI})$, show that $f(\mathbf{I}) = 1$.
- (ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $f(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $f(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d ,

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -f\begin{pmatrix} c & d \\ a & b \end{pmatrix} = f\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

- (iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $f(\mathbf{A}) = 0$.
- (iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2$, $\mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{P}\mathbf{K}$ for suitable values of k , evaluate $f(\mathbf{P})$.

The hook

The determinant of a 2×2 matrix is $ad - bc$. Where does that formula come from, and why is it so important? One answer is that it is the factor by which the matrix scales areas. Another, which this question explores, is that the determinant is essentially *the only* function on matrices that turns matrix multiplication into ordinary multiplication: $\det(\mathbf{MN}) = \det \mathbf{M} \det \mathbf{N}$.

The question gives you an unknown function f with that one property (plus two mild conditions to rule out trivial cases), and asks you to deduce, step by step, how it must behave. Like the derivations of Section 6.5, it is an exercise in pinning an object down by its properties alone. By the end, f has been forced to change sign when rows are swapped, vanish on singular matrices and equal 1 on shears: exactly the behaviour of

the determinant.

Before you start

Ask yourself What does each special matrix in the question do? Work out $\mathbf{JM}$, $\mathbf{MJ}$, $\mathbf{KM}$ and $\mathbf{PM}$ for a general $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\mathbf{JM}$ swaps the rows of $\mathbf{M}$; $\mathbf{MJ}$ swaps its columns. $\mathbf{KM}$ multiplies the second row by k . $\mathbf{PM}$ adds the second row to the first. These are the *elementary row operations* of Gaussian elimination. Since f turns products into products, knowing f on these special matrices tells you how f responds to each row operation.

Working it through

STAGE 1 The identity

$f(\mathbf{M}) = f(\mathbf{MI}) = f(\mathbf{M})f(\mathbf{I})$. Since $f(\mathbf{M}) \neq 0$, divide to get $f(\mathbf{I}) = 1$. ■

STAGE 2 Swapping

$\mathbf{J}^2 = \mathbf{I}$, so $f(\mathbf{J})^2 = f(\mathbf{I}) = 1$, giving $f(\mathbf{J}) = \pm 1$. Since $f(\mathbf{J}) \neq 1$, $f(\mathbf{J}) = -1$.

Now $\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \mathbf{JM}$, so its value is $f(\mathbf{J})f(\mathbf{M}) = -f(\mathbf{M})$. And $\begin{pmatrix} d & c \\ b & a \end{pmatrix} = \mathbf{MJ}$ (swap rows, then columns), so its value is $f(\mathbf{J})^2f(\mathbf{M}) = f(\mathbf{M})$. ■

STAGE 3 Dependent rows

Ask yourself First consider a matrix $\mathbf{B}$ with two *equal* rows. What is $\mathbf{JB}$? What does that say about $f(\mathbf{B})$?

If both rows of $\mathbf{B}$ are (a, b) , swapping them changes nothing: $\mathbf{JB} = \mathbf{B}$. So $f(\mathbf{B}) = f(\mathbf{J})f(\mathbf{B}) = -f(\mathbf{B})$, which forces $f(\mathbf{B}) = 0$.

Now if $\mathbf{A}$ has rows (a, b) and $k(a, b)$, then $\mathbf{A} = \mathbf{KB}$, where $\mathbf{B}$ has both rows equal to (a, b) . So $f(\mathbf{A}) = f(\mathbf{K})f(\mathbf{B}) = 0$. ■ (This includes $k = 0$, where the second row is zero.)

STAGE 4 The shear

Pause and try Compute $\mathbf{P}^2$ and $\mathbf{K}^{-1}\mathbf{PK}$. Choose k to make them equal. What does that give for $f(\mathbf{P})$?

$\mathbf{P}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and, for $k \neq 0$,

$$\mathbf{K}^{-1}\mathbf{P}\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

With $k = 2$, $\mathbf{K}^{-1}\mathbf{P}\mathbf{K} = \mathbf{P}^2$. Taking f of both sides, and using $f(\mathbf{K}^{-1})f(\mathbf{K}) = f(\mathbf{I}) = 1$,

$$f(\mathbf{P}) = f(\mathbf{P})^2,$$

so $f(\mathbf{P}) = 0$ or 1 . But $\mathbf{P}$ is invertible, and $f(\mathbf{P})f(\mathbf{P}^{-1}) = f(\mathbf{I}) = 1$, so $f(\mathbf{P}) \neq 0$. Therefore $f(\mathbf{P}) = 1$. ■

Looking back

- **Special elements reveal the function.** Applying the multiplicative property to $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{P}$ gave the value of f on each, and its effect on each row operation.
- **An element equal to its own transform.** A matrix with equal rows is unchanged by a row swap, which changes the sign of f ; so f must be zero there.
- **Conjugation preserves f .** $f(\mathbf{K}^{-1}\mathbf{P}\mathbf{K}) = f(\mathbf{P})$, a fact used again and again in algebra.

What's really going on

Every multiplicative function is a function of the determinant

The question stops at $f(\mathbf{P}) = 1$, but the argument can be completed. Every invertible matrix is a product of elementary matrices (row swaps, row scalings and shears), and the question shows how f behaves on each. It follows that $f(\mathbf{M}) = \phi(\det \mathbf{M})$ for some function ϕ of a single real number with $\phi(xy) = \phi(x)\phi(y)$. The determinant itself is the case $\phi(x) = x$, but $\phi(x) = x^3$, or $\phi(x) = |x|^{1/2} \text{sign}(x)$, work too. In other words, *the determinant is the only fundamentally new multiplicative function on matrices*. In the language of group theory, the determinant captures everything about the group of invertible matrices that can be seen by a commutative group.

The determinant as area

Geometrically, a 2×2 matrix sends the unit square to a parallelogram, and $\det \mathbf{M}$ is its *signed area*: positive if the orientation is preserved, negative if it is reversed (Figure 7.3.1). The multiplicative property then has an obvious meaning: doing two transformations in turn multiplies their area scale factors. A row swap reverses orientation, which is why $\det \mathbf{J} = -1$; a shear slides the square into a parallelogram of the same area, which is why $\det \mathbf{P} = 1$; and a matrix with dependent rows squashes the square flat, which is why its determinant is zero. Every step of the question has a picture.

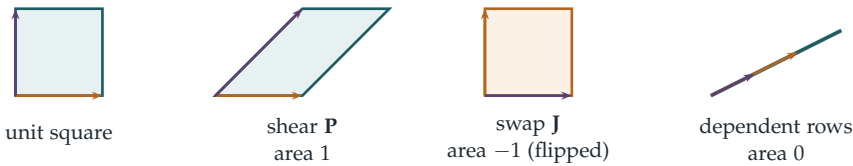


Figure 7.3.1 The determinant as signed area. The images of the two basis vectors (amber, plum) span a parallelogram. A shear preserves area; a swap of the basis vectors reverses orientation; dependent rows (or columns) collapse the square onto a line. Doing two transformations in turn multiplies the signed areas, which is the property $f(\mathbf{MN}) = f(\mathbf{M})f(\mathbf{N})$.

Determinants in higher dimensions

For $n \times n$ matrices the determinant is the signed volume scale factor, and it is characterised by three properties: it is linear in each row, it changes sign when two rows are swapped, and it equals 1 on the identity. From these alone you can derive the familiar expansion formulas, and Gaussian elimination gives an efficient way to compute it. The determinant appears in the change-of-variables formula for multiple integrals (the *Jacobian*), in the formula for the inverse of a matrix, and, as we saw in Section 5.4, in the discriminant of a polynomial.

The thinking skill

Probe an unknown function with special inputs

When a function is known only through a property, apply the property to the simplest special inputs you can think of, and see what it forces.

Here the identity, a swap, a scaling and a shear were enough to determine almost everything. Choosing inputs that are their own squares, inverses or conjugates is particularly effective.

Try this next

- Other examples.** Show that $f(\mathbf{M}) = (\det \mathbf{M})^3$ satisfies all the conditions of the question, and that $f(\mathbf{M}) = |\det \mathbf{M}|$ satisfies the multiplicative property but not $f(\mathbf{J}) \neq 1$.
- Column operations.** Show that f is also zero on any matrix whose second column is a multiple of the first.
- Scalings.** Let $\mathbf{K}_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. Show that $f\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right) = f(\mathbf{K}_k)$, and that $f(\mathbf{K}_k)f(\mathbf{K}_l) = f(\mathbf{K}_{kl})$.

- 4 Reduction to the determinant.** **HARDER** Show that every invertible 2×2 matrix can be written as a product of matrices of the forms $\mathbf{J}$, $\mathbf{K}_k$ and $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Deduce that $f(\mathbf{M})$ depends only on $\det \mathbf{M}$.

Further reading

- **Sheldon Axler**, *Linear Algebra Done Right* (Springer; free online). Defines the determinant from its properties, in the spirit of this question.
- **Gilbert Strang**, *Introduction to Linear Algebra* (Wellesley-Cambridge). Chapter 5 develops determinants from three rules.
- **3Blue1Brown**, “The determinant” (video, free online). The area interpretation, beautifully animated.

7.4 Matrices That Return to the Start

The question

A 2×2 matrix $\mathbf{M}$ is real if it can be written as $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real. In this case, the *trace* of matrix $\mathbf{M}$ is defined to be $\text{tr}(\mathbf{M}) = a + d$ and $\det(\mathbf{M})$ is the determinant of matrix $\mathbf{M}$. In this question, $\mathbf{M}$ is a real 2×2 matrix.

(i) Prove that

$$\text{tr}(\mathbf{M}^2) = \text{tr}(\mathbf{M})^2 - 2 \det(\mathbf{M}).$$

(ii) Prove that

$$\mathbf{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm \mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1,$$

and that

$$\mathbf{M}^2 = -\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part (ii) to prove that

$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm \mathbf{I}.$$

Find a necessary and sufficient condition on $\det(\mathbf{M})$ and $\text{tr}(\mathbf{M})$ so that $\mathbf{M}^4 = -\mathbf{I}$.

(iv) Give an example of a matrix $\mathbf{M}$ for which $\mathbf{M}^8 = \mathbf{I}$, but which does not represent a rotation or reflection. [Note that the matrices $\pm \mathbf{I}$ are both rotations.]

The hook

Some transformations, repeated enough times, bring everything back to where it started. A rotation by 90° does it in four steps, a reflection in two. Which 2×2 matrices have this property, $\mathbf{M}^n = \mathbf{I}$ for some n , and how can you tell?

The surprise of this question is that two numbers, the *trace* and the *determinant*, are enough to decide. You do not need to know the four entries, only $a + d$ and $ad - bc$. And the answer has a striking consequence for the physical world: it explains why crystals can have 2-, 3-, 4- and 6-fold symmetry but never 5-fold, a “law” of crystallography that held for two centuries until a discovery in 1982 won a Nobel Prize.

Before you start



Ask yourself Every 2×2 matrix satisfies $\mathbf{M}^2 - (\text{tr } \mathbf{M})\mathbf{M} + (\det \mathbf{M})\mathbf{I} = \mathbf{0}$ (the Cayley–Hamilton theorem; check it by multiplying out). How could that help with questions about $\mathbf{M}^2$?

It expresses $\mathbf{M}^2$ in terms of $\mathbf{M}$ and $\mathbf{I}$: $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I}$, where $t = \text{tr } \mathbf{M}$ and $d = \det \mathbf{M}$. So a condition on $\mathbf{M}^2$ becomes a condition linking $\mathbf{M}$, t and d . That is the key to the whole question.

Working it through

STAGE 1 The trace of the square

$$\mathbf{M}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}, \text{ so } \text{tr}(\mathbf{M}^2) = a^2 + 2bc + d^2 = (a + d)^2 - 2(ad - bc). \quad \blacksquare$$

STAGE 2 Square roots of $\pm \mathbf{I}$



Ask yourself Suppose $\mathbf{M}^2 = \mathbf{I}$. Substitute into $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I}$. If $t \neq 0$, what does that say about $\mathbf{M}$?

If $\mathbf{M}^2 = \mathbf{I}$, then $\mathbf{I} = t\mathbf{M} - d\mathbf{I}$, so $t\mathbf{M} = (1 + d)\mathbf{I}$. If $t \neq 0$, then $\mathbf{M}$ is a multiple of $\mathbf{I}$, say $c\mathbf{I}$, and $c^2 = 1$, so $\mathbf{M} = \pm \mathbf{I}$, which is excluded. Hence $t = 0$, and then $(1 + d)\mathbf{I} = \mathbf{0}$, so $d = -1$.

Conversely, if $t = 0$ and $d = -1$, then $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I} = \mathbf{I}$, and $\mathbf{M} \neq \pm \mathbf{I}$ because $\text{tr}(\pm \mathbf{I}) = \pm 2 \neq 0$. ■

For $\mathbf{M}^2 = -\mathbf{I}$: substituting gives $t\mathbf{M} = (d - 1)\mathbf{I}$. If $t \neq 0$ then $\mathbf{M} = c\mathbf{I}$ with $c^2 = -1$, impossible for real c . So $t = 0$ and $d = 1$. Conversely, $t = 0$ and $d = 1$ give $\mathbf{M}^2 = -\mathbf{I}$. ■

STAGE 3 Fourth roots



Pause and try Let $\mathbf{N} = \mathbf{M}^2$, so $\mathbf{M}^4 = \mathbf{I}$ means $\mathbf{N}^2 = \mathbf{I}$. By part (ii), either $\mathbf{N} = \pm \mathbf{I}$ or $\text{tr } \mathbf{N} = 0$ and $\det \mathbf{N} = -1$. Why is the second case impossible?

$\det \mathbf{N} = \det(\mathbf{M}^2) = (\det \mathbf{M})^2 \geq 0$, so $\det \mathbf{N} \neq -1$. Hence $\mathbf{N} = \mathbf{M}^2 = \pm \mathbf{I}$. The converse is clear. ■

For $\mathbf{M}^4 = -\mathbf{I}$: by part (ii) applied to $\mathbf{N} = \mathbf{M}^2$, this holds exactly when $\text{tr}(\mathbf{M}^2) = 0$ and $\det(\mathbf{M}^2) = 1$, that is, $t^2 - 2d = 0$ and $d^2 = 1$. Since $t^2 = 2d \geq 0$, we need $d = 1$, and then $t^2 = 2$. So

$$\mathbf{M}^4 = -\mathbf{I} \iff \det \mathbf{M} = 1 \text{ and } \text{tr } \mathbf{M} = \pm\sqrt{2}. \quad \blacksquare$$

STAGE 4 An eighth root of $\mathbf{I}$ that is not a rotation

Any matrix with $\mathbf{M}^4 = -\mathbf{I}$ has $\mathbf{M}^8 = \mathbf{I}$. A rotation by 45° has trace $\sqrt{2}$ and determinant 1; we want another matrix with the same trace and determinant that is not a rotation. Take

$$\mathbf{M} = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{tr } \mathbf{M} = \sqrt{2}, \quad \det \mathbf{M} = 1.$$

By Stage 3, $\mathbf{M}^4 = -\mathbf{I}$, so $\mathbf{M}^8 = \mathbf{I}$. But $\mathbf{M}$ is not a rotation or reflection, because those preserve lengths, and $\mathbf{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ has length $\sqrt{3} \neq 1$. ■

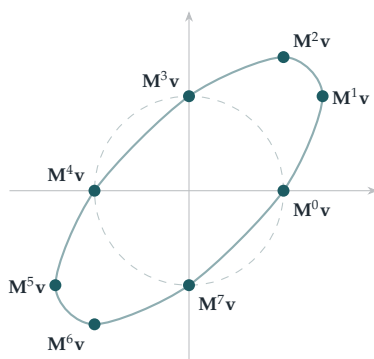


Figure 7.4.1 The orbit of $\mathbf{v} = (1, 0)$ under $\mathbf{M} = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$. After eight steps it returns to the start. The points do not lie on a circle (dashed), so $\mathbf{M}$ is not a rotation, but they do lie on an ellipse: $\mathbf{M}$ is a 45° rotation viewed in skewed coordinates.

Looking back

- **Cayley–Hamilton turns powers into linear combinations.** Every power of a 2×2 matrix is a combination of $\mathbf{M}$ and $\mathbf{I}$, with coefficients depending only on trace and determinant.
- **Determinants of squares are non-negative.** That simple fact ruled out a whole case in part (iii).
- **Trace and determinant are invariants.** Two matrices with the same trace and determinant (and not multiples of $\mathbf{I}$) behave alike under powers, whether or not they look alike.

What's really going on

Similar matrices

Why is $\mathbf{M}$ in Stage 4 “a rotation in disguise”? Two matrices $\mathbf{M}$ and $\mathbf{R}$ are *similar* if $\mathbf{M} = \mathbf{SRS}^{-1}$ for some invertible $\mathbf{S}$: they represent the same transformation in different coordinates. Similar matrices have the same trace and determinant, and (for 2×2 matrices that are not multiples of $\mathbf{I}$) the converse holds too. So any real matrix with trace $\sqrt{2}$ and determinant 1 is similar to the rotation by 45° , and in the skewed coordinates given by $\mathbf{S}$ it simply rotates (Figure 7.4.1). In general, a real 2×2 matrix with $\mathbf{M}^n = \mathbf{I}$ is always similar to a rotation or a reflection.

Why crystals have no five-fold symmetry

A crystal is a regular, repeating arrangement of atoms. A rotational symmetry of the crystal sends the lattice of atoms to itself, so in coordinates based on the lattice it is given by a matrix with *integer* entries. Its trace is therefore an integer. But a rotation by angle θ has trace $2 \cos \theta$ in any coordinates, and $2 \cos \theta$ is an integer only when it is $-2, -1, 0, 1$ or 2 , that is, for rotations of order 2, 3, 4 and 6 (and 1). Five-fold symmetry would need $2 \cos 72^\circ = \frac{\sqrt{5}-1}{2}$, which is not an integer. This is the *crystallographic restriction theorem*. In 1982 Dan Shechtman observed an alloy with apparent five-fold symmetry and was at first ridiculed; it turned out to be a *quasicrystal*, ordered but not periodic, like the Penrose tilings, and so not covered by the theorem. He received the Nobel Prize in Chemistry in 2011.

The thinking skill

Look for invariants that capture the essential behaviour

When a question is about an object with many parameters, ask whether a few combinations of them, invariant under a change of viewpoint, are enough.

Here four entries reduced to two invariants, the trace and the determinant, because the question (“does $\mathbf{M}^n = \mathbf{I}$?”) does not depend on the coordinates. Finding the right invariants is often the key to classification problems.

Try this next

- 1 Rotations of order three.** (From STEP II, 2021.) The matrix $\mathbf{R}$ represents an anti-clockwise rotation through angle ϕ , and $\mathbf{R} + \mathbf{I}$ also represents a rotation. Find the possible values of ϕ and deduce that $\mathbf{R}^3 = \mathbf{I}$. Then show that if $\mathbf{S}$ is a real matrix with $\mathbf{S}^3 = \mathbf{I}$ but $\mathbf{S} \neq \mathbf{I}$, then $\det \mathbf{S} = 1$ and $\text{tr } \mathbf{S} = -1$.

- 2 Order six.** Find a condition on the trace and determinant for $\mathbf{M}^6 = \mathbf{I}$ but $\mathbf{M}^2 \neq \mathbf{I}$ and $\mathbf{M}^3 \neq \mathbf{I}$.
- 3 The crystallographic restriction.** Show that if $\mathbf{M}$ has integer entries, $\det \mathbf{M} = 1$ and $\mathbf{M}^n = \mathbf{I}$ for some n , then $\operatorname{tr} \mathbf{M} \in \{-2, -1, 0, 1, 2\}$. Find an integer matrix of order 6.
- 4 Cayley–Hamilton for powers.** Use $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I}$ to express $\mathbf{M}^3$ and $\mathbf{M}^4$ in the form $p\mathbf{M} + q\mathbf{I}$, and use this to rederive the condition for $\mathbf{M}^4 = -\mathbf{I}$.

Further reading

- **M. A. Armstrong, *Groups and Symmetry*** (Springer). Includes the crystallographic restriction and the classification of wallpaper patterns.
- **Marjorie Senechal, *Quasicrystals and Geometry*** (Cambridge University Press). The mathematics of quasicrystals and aperiodic order.
- **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge). Similar matrices and the Cayley–Hamilton theorem.

7.5 Rotating in Three Dimensions

The question

Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $\mathbf{f}(\mathbf{r})$ is defined by $\mathbf{f}(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

(i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the x -component of $\mathbf{f}(\mathbf{f}(\mathbf{r}))$ is $-x(b^2 + c^2) + aby + acz$. Show further that

$$\mathbf{f}(\mathbf{f}(\mathbf{r})) = (\mathbf{n} \cdot \mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $\mathbf{f}(\mathbf{f}(\mathbf{r}))$ is related to $\mathbf{n}$ and $\mathbf{r}$.

(ii) Let R be the point with position vector $\mathbf{r}$ and P be the point with position vector $\mathbf{g}(\mathbf{r})$, where $\mathbf{g}$ is defined by

$$\mathbf{g}(\mathbf{s}) = \mathbf{s} + \sin \theta \mathbf{f}(\mathbf{s}) + (1 - \cos \theta) \mathbf{f}(\mathbf{f}(\mathbf{s})).$$

By considering $\mathbf{g}(\mathbf{n})$ and $\mathbf{g}(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps R onto P .

(iii) Let R be the point with position vector $\mathbf{r}$ and Q be the point with position vector $\mathbf{h}(\mathbf{r})$, where $\mathbf{h}$ is defined by

$$\mathbf{h}(\mathbf{s}) = -\mathbf{s} - 2\mathbf{f}(\mathbf{f}(\mathbf{s})).$$

State, with justification, the geometric transformation which maps R onto Q .

The hook

In two dimensions, rotating a point is easy: multiply by $e^{i\theta}$. In three dimensions it is much harder. There is no single complex number to multiply by, rotations about different axes do not commute, and writing down the matrix for “rotate by 37° about the axis through $(1, 2, 2)$ ” looks daunting.

This question builds that matrix, for any axis and any angle, out of the cross product. The result is *Rodrigues’ rotation formula*, published by Olinde Rodrigues in 1840. It is used every day in robotics, computer graphics and spacecraft navigation, and it reveals

a lovely fact: crossing with $\mathbf{n}$ is “rotation by 90° about $\mathbf{n}$ ”, and every rotation about $\mathbf{n}$ is built from it, just as $e^{i\theta} = \cos \theta + i \sin \theta$ is built from multiplication by i .

Before you start



Ask yourself What does $\mathbf{n} \times \mathbf{r}$ look like geometrically, when $\mathbf{r}$ is perpendicular to $\mathbf{n}$? And when $\mathbf{r}$ is parallel to $\mathbf{n}$?

If $\mathbf{r} \parallel \mathbf{n}$, then $\mathbf{n} \times \mathbf{r} = \mathbf{0}$. If $\mathbf{r} \perp \mathbf{n}$, then $\mathbf{n} \times \mathbf{r}$ has the same length as $\mathbf{r}$ (since $|\mathbf{n}| = 1$ and the angle is 90°), is perpendicular to both, and by the right-hand rule is $\mathbf{r}$ rotated by 90° anticlockwise about $\mathbf{n}$ (looking down $\mathbf{n}$). So f kills the component of $\mathbf{r}$ along $\mathbf{n}$ and rotates the perpendicular component by 90° . That picture explains the whole question.

Working it through

STAGE 1 Crossing twice

$f(\mathbf{r}) = \mathbf{n} \times \mathbf{r} = (bz - cy, cx - az, ay - bx)$. Crossing again with $\mathbf{n} = (a, b, c)$, the x -component of $f(f(\mathbf{r}))$ is

$$b(ay - bx) - c(cx - az) = -x(b^2 + c^2) + aby + acz. \quad \blacksquare$$

Since $a^2 + b^2 + c^2 = 1$, this is $-x(1 - a^2) + aby + acz = a(ax + by + cz) - x$, which is the x -component of $(\mathbf{n} \cdot \mathbf{r})\mathbf{n} - \mathbf{r}$. The other components follow by symmetry, so $f(f(\mathbf{r})) = (\mathbf{n} \cdot \mathbf{r})\mathbf{n} - \mathbf{r}$. $\blacksquare$



Ask yourself Write $\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$, where $\mathbf{r}_{\parallel} = (\mathbf{n} \cdot \mathbf{r})\mathbf{n}$ is the component along $\mathbf{n}$. What is $f(f(\mathbf{r}))$ in terms of these?

$f(f(\mathbf{r})) = \mathbf{r}_{\parallel} - \mathbf{r} = -\mathbf{r}_{\perp}$. Applying f twice kills the parallel part and rotates the perpendicular part by 180° about $\mathbf{n}$, which simply reverses it (Figure 7.5.1). This is the analogue of $i^2 = -1$.

STAGE 2 The rotation



Ask yourself Compute $g(\mathbf{n})$. Then compute $g(\mathbf{r})$ for $\mathbf{r} \perp \mathbf{n}$, using $f(f(\mathbf{r})) = -\mathbf{r}$ for such $\mathbf{r}$.

$f(\mathbf{n}) = \mathbf{n} \times \mathbf{n} = \mathbf{0}$, so $g(\mathbf{n}) = \mathbf{n}$: the axis is fixed. If $\mathbf{r} \perp \mathbf{n}$, then $f(f(\mathbf{r})) = -\mathbf{r}$ and

$$g(\mathbf{r}) = \mathbf{r} + \sin \theta (\mathbf{n} \times \mathbf{r}) - (1 - \cos \theta)\mathbf{r} = \cos \theta \mathbf{r} + \sin \theta (\mathbf{n} \times \mathbf{r}).$$

Since $\mathbf{n} \times \mathbf{r}$ is $\mathbf{r}$ turned through 90° about $\mathbf{n}$, with the same length, this is $\mathbf{r}$ rotated through θ about $\mathbf{n}$, exactly like $(\cos \theta + i \sin \theta)z$ in the plane. Since g is linear and every vector is a sum of a multiple of $\mathbf{n}$ and a vector perpendicular to $\mathbf{n}$, g is the **rotation**

through angle θ about the axis through the origin in the direction $\mathbf{n}$ (anticlockwise when viewed from the tip of $\mathbf{n}$). ■

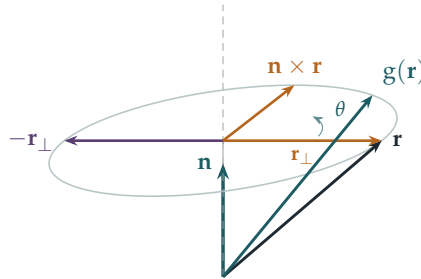


Figure 7.5.1 Rodrigues' formula. The vector $\mathbf{r}$ splits into a part along the axis $\mathbf{n}$ and a part $\mathbf{r}_\perp$ perpendicular to it, which sweeps out a circle. Crossing with $\mathbf{n}$ turns $\mathbf{r}_\perp$ through 90° (amber); crossing twice gives $-\mathbf{r}_\perp$ (plum). The rotated vector $\mathbf{g}(\mathbf{r})$ is $\mathbf{r}_\parallel + \cos \theta \mathbf{r}_\perp + \sin \theta (\mathbf{n} \times \mathbf{r})$.

STAGE 3 The reflection



Pause and try Simplify $\mathbf{h}(\mathbf{s}) = -\mathbf{s} - 2\mathbf{f}(\mathbf{f}(\mathbf{s}))$ using part (i). What does it do to vectors along $\mathbf{n}$, and to vectors perpendicular to $\mathbf{n}$?

$\mathbf{h}(\mathbf{s}) = -\mathbf{s} - 2((\mathbf{n} \cdot \mathbf{s})\mathbf{n} - \mathbf{s}) = \mathbf{s} - 2(\mathbf{n} \cdot \mathbf{s})\mathbf{n}$. Vectors perpendicular to $\mathbf{n}$ are unchanged, and $\mathbf{h}(\mathbf{n}) = -\mathbf{n}$. So $\mathbf{h}$ is the **reflection in the plane through the origin perpendicular to $\mathbf{n}$** . ■

Looking back

- **Decompose into parallel and perpendicular parts.** Every step became clear once $\mathbf{r}$ was split into its components along and across $\mathbf{n}$.
- **Linearity reduces the work.** To identify a linear map, it is enough to know what it does to a few well-chosen vectors.
- **An analogue of complex numbers.** On vectors perpendicular to $\mathbf{n}$, $\mathbf{f}$ behaves like multiplication by i : $\mathbf{f}^2 = -1$, and $\mathbf{g} = \cos \theta + \sin \theta \mathbf{f}$.

What's really going on

Euler's rotation theorem

Every rotation of three-dimensional space that fixes the origin is a rotation about some axis through the origin. Euler proved this in 1775. In matrix terms, every 3×3 rotation matrix has an eigenvector with eigenvalue 1, the axis. Rodrigues' formula gives the

converse: for every axis $\mathbf{n}$ and angle θ , the rotation matrix is

$$\mathbf{R} = \mathbf{I} + \sin \theta \mathbf{K} + (1 - \cos \theta) \mathbf{K}^2, \quad \mathbf{K} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

where $\mathbf{K}$ is the matrix of $\mathbf{r} \mapsto \mathbf{n} \times \mathbf{r}$. Since $\mathbf{K}^3 = -\mathbf{K}$, the exponential series $e^{\theta \mathbf{K}} = \mathbf{I} + \theta \mathbf{K} + \frac{\theta^2}{2!} \mathbf{K}^2 + \cdots$ collapses to exactly this formula, the three-dimensional version of $e^{i\theta} = \cos \theta + i \sin \theta$. This connection between rotations and exponentials of “infinitesimal rotations” $\mathbf{K}$ is the starting point of the theory of *Lie groups*.

Rotations do not commute

Hold a book flat, rotate it 90° about a vertical axis and then 90° about a horizontal axis; then start again and do the two rotations in the other order. The book ends up in different positions. Composition of rotations in three dimensions is not commutative, unlike in two dimensions. This is one reason the quaternions of Section 6.7 are useful: the rotation by θ about $\mathbf{n}$ corresponds to the quaternion $\cos \frac{\theta}{2} + \sin \frac{\theta}{2}(ai + bj + ck)$, and composing rotations is multiplying quaternions, with the non-commutativity built in. Rodrigues himself, in his 1840 paper, found the composition formula for rotations three years before Hamilton invented quaternions.

The thinking skill

Split into invariant parts

To understand a transformation, split space into pieces it treats simply: here, the axis (left alone) and the plane perpendicular to it (rotated).

This is the same idea as eigenvectors in Section 7.2: find subspaces the transformation preserves, and study it on each piece.

Try this next

- 1 Rotations in the plane.** (From STEP III, 2017.) Show that rotation by θ about the point a in the complex plane sends z to $e^{i\theta}z + a(1 - e^{i\theta})$. Show that rotating by θ about a and then by ϕ about b is a rotation by $\theta + \phi$ about some point c , provided $\theta + \phi$ is not a multiple of 2π . What is it when $\theta + \phi = 2\pi$?
- 2 A specific rotation.** Find the matrix of the rotation by 120° about the axis in the direction $(1, 1, 1)$. Show that it permutes the coordinate axes cyclically.

- 3 Powers of $\mathbf{K}$.** Show that $\mathbf{K}^2\mathbf{r} = (\mathbf{n}\cdot\mathbf{r})\mathbf{n} - \mathbf{r}$ and $\mathbf{K}^3 = -\mathbf{K}$, and deduce Rodrigues' formula from the exponential series.
- 4 Two reflections make a rotation.** Show that reflecting in two planes through the origin, at angle α to each other, is a rotation by 2α about their line of intersection.

Further reading

- **Simon Altmann, *Rotations, Quaternions, and Double Groups*** (Dover). The definitive account of three-dimensional rotations, including Rodrigues' contribution.
- **Simon Altmann, "Hamilton, Rodrigues, and the quaternion scandal", *Mathematics Magazine* 62 (1989).** A short, lively history.
- **John Stillwell, *Naive Lie Theory*** (Springer). Rotations, quaternions and Lie groups for undergraduates.

7.6 Hyperbolic Rotations and Relativity

The question

The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where u is a positive real constant. Show that the curve with equation $x^2 - y^2 = 1$ is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves.

A line, not necessarily through the origin, which has gradient $\tanh v$ transforms under T into a line with gradient $\tanh v'$. Show that $v' = v + u$.

The lines ℓ_1 and ℓ_2 with gradients $\tanh v_1$ and $\tanh v_2$ transform under T into lines with gradients $\tanh v'_1$ and $\tanh v'_2$ respectively. Find the relation satisfied by v_1 and v_2 that is necessary and sufficient for ℓ_1 and ℓ_2 to intersect at the same angle as their transforms.

In the case when ℓ_1 and ℓ_2 meet at the origin, illustrate in a diagram the relation between ℓ_1 , ℓ_2 and their transforms.

The hook

A rotation matrix $\begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$ moves points round the circle $x^2 + y^2 = 1$, and rotating by u then by w is rotating by $u + w$. Replace $\cos$ and $\sin$ by $\cosh$ and $\sinh$, adjust a sign, and you get the matrix in this question. It moves points along the hyperbola $x^2 - y^2 = 1$ instead, and it too adds its “angles”. It is a *hyperbolic rotation*.

This is not just a curiosity. With x standing for time and y for distance (in units where the speed of light is 1), the matrix is a *Lorentz boost*: the transformation, discovered in Einstein’s special relativity, between the measurements of two observers moving at constant velocity relative to each other. The “hyperbolic angle” u is the *rapidity*, and the fact that rapidities add explains the strange velocity-addition law we met in Section 3.8.

Before you start



Ask yourself What identity do $\cosh u$ and $\sinh u$ satisfy, analogous to $\cos^2 u + \sin^2 u = 1$? And what are the addition formulas for $\cosh(u + v)$ and $\sinh(u + v)$?

$\cosh^2 u - \sinh^2 u = 1$, and

$$\cosh(u + v) = \cosh u \cosh v + \sinh u \sinh v,$$

$$\sinh(u + v) = \sinh u \cosh v + \cosh u \sinh v.$$

You can check these from the definitions $\cosh u = \frac{1}{2}(e^u + e^{-u})$ and $\sinh u = \frac{1}{2}(e^u - e^{-u})$. The point $(\cosh v, \sinh v)$ lies on the hyperbola $x^2 - y^2 = 1$, just as $(\cos v, \sin v)$ lies on the circle.

Working it through

STAGE 1 The hyperbola is invariant

$$\begin{aligned} x'^2 - y'^2 &= (x \cosh u + y \sinh u)^2 - (x \sinh u + y \cosh u)^2 \\ &= (x^2 - y^2)(\cosh^2 u - \sinh^2 u) = x^2 - y^2, \end{aligned}$$

since the cross terms $2xy \cosh u \sinh u$ cancel. So T preserves $x^2 - y^2$, and in particular maps the hyperbola $x^2 - y^2 = 1$ to itself. ■



Ask yourself Which lines through the origin are mapped to themselves? (Look for eigenvectors.)

$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\cosh u + \sinh u) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^u \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-u} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. So the lines $y = x$ and $y = -x$ are each mapped to themselves: one is stretched by e^u , the other shrunk by e^{-u} . These are the asymptotes of the hyperbola. ■

STAGE 2 Adding hyperbolic angles

A line with gradient $\tanh v$ has direction vector $(\cosh v, \sinh v)$. Its image has direction

$$T \begin{pmatrix} \cosh v \\ \sinh v \end{pmatrix} = \begin{pmatrix} \cosh u \cosh v + \sinh u \sinh v \\ \sinh u \cosh v + \cosh u \sinh v \end{pmatrix} = \begin{pmatrix} \cosh(u + v) \\ \sinh(u + v) \end{pmatrix},$$

which has gradient $\tanh(u + v)$. A linear map sends parallel lines to parallel lines, so this holds whether or not the line passes through the origin. Hence $v' = v + u$. ■

STAGE 3 When are angles preserved?

Ask yourself The angle ψ between two lines with gradients m_1, m_2 satisfies $\tan \psi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$. Simplify this when $m_i = \tanh v_i$. (*Hint: put everything over $\cosh v_1 \cosh v_2$.*)

Using $\tanh v_1 - \tanh v_2 = \frac{\sinh(v_1 - v_2)}{\cosh v_1 \cosh v_2}$ and $1 + \tanh v_1 \tanh v_2 = \frac{\cosh(v_1 + v_2)}{\cosh v_1 \cosh v_2}$,

$$\tan \psi = \left| \frac{\sinh(v_1 - v_2)}{\cosh(v_1 + v_2)} \right|.$$

After the transformation, $v_1 - v_2$ is unchanged and $v_1 + v_2$ becomes $v_1 + v_2 + 2u$.

 **Pause and try** When is $\cosh(v_1 + v_2) = \cosh(v_1 + v_2 + 2u)$?

Assuming the lines are not parallel ($v_1 \neq v_2$), the angles are equal exactly when $\cosh(v_1 + v_2) = \cosh(v_1 + v_2 + 2u)$. Since $\cosh$ is even and strictly increasing on $[0, \infty)$, this means $v_1 + v_2 = \pm(v_1 + v_2 + 2u)$. The plus sign gives $u = 0$, which is excluded, so

$$v_1 + v_2 = -u. \quad \blacksquare$$

STAGE 4 The picture

When $v_1 + v_2 = -u$, the transformed angles are $v'_1 = v_1 + u = -v_2$ and $v'_2 = v_2 + u = -v_1$. So the image of ℓ_1 has gradient $-\tanh v_2$: it is the reflection of ℓ_2 in the x -axis, and the image of ℓ_2 is the reflection of ℓ_1 . For lines through the origin, the transformation swaps the two lines and reflects them, which explains why the angle between them is unchanged (Figure 7.6.1).

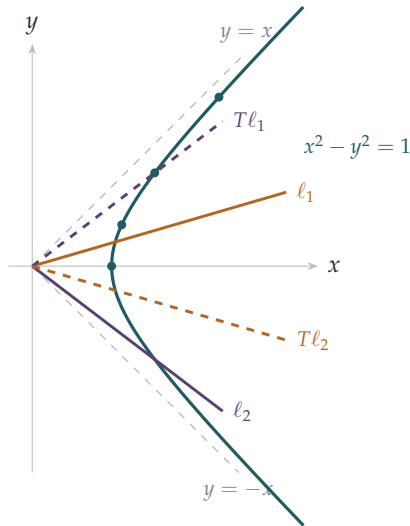


Figure 7.6.1 A hyperbolic rotation with $u = 0.7$ applied to lines ℓ_1, ℓ_2 through the origin with $v_1 = 0.3, v_2 = -1.0$, so that $v_1 + v_2 = -u$. The image of ℓ_1 (dashed plum) is the mirror image of ℓ_2 , and the image of ℓ_2 (dashed amber) is the mirror image of ℓ_1 , so the angle between the lines is preserved. The teal dots on the hyperbola are equally spaced in hyperbolic angle ($v = 0, 0.5, 1, 1.5$); T moves each one to the next point 0.7 further along.

Looking back

- **An invariant quadratic form.** T preserves $x^2 - y^2$, just as a rotation preserves $x^2 + y^2$.
- **Eigenvectors are the asymptotes.** The lines left invariant are $y = \pm x$, stretched and shrunk by $e^{\pm u}$.
- **Hyperbolic angles add.** Parametrising directions by $\tanh v$ turned the action of T into simple addition, $v \mapsto v + u$.

What's really going on

Special relativity

In Einstein's special relativity (1905), different observers moving at constant velocity relative to each other disagree about time and distance, but they all agree on the quantity $t^2 - x^2$ (with the speed of light $c = 1$), the *spacetime interval*. The transformations preserving it are the *Lorentz transformations*, and in one space dimension they are exactly the matrices in this question, with x as time and y as distance. An observer moving with velocity $v = \tanh u$ uses coordinates related to ours by this matrix, with $\cosh u = \frac{1}{\sqrt{1-v^2}}$, the famous factor γ of time dilation. Hermann Minkowski saw in

1908 that this makes spacetime a four-dimensional geometry with a “distance” that has a minus sign in it: “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows.”

What the question means physically

- **The invariant lines** $y = \pm x$ are the paths of light rays. They are the same for every observer: the speed of light is the same for everyone, the founding principle of relativity.
- **The hyperbola** $x^2 - y^2 = 1$ is the set of events one unit of “proper time” from the origin, as measured by observers moving at all possible constant velocities.
- **Adding rapidities.** A line of gradient $\tanh v$ through the origin is the path of an object moving at velocity $\tanh v$. A boost by rapidity u changes its velocity to $\tanh(v + u) = \frac{\tanh v + \tanh u}{1 + \tanh v \tanh u}$, which is Einstein’s velocity-addition formula from Section 3.8. Velocities do not add; rapidities do.

And in Section 5.8 we saw the same addition formula as the distance rule in the Poincaré disc: the set of possible velocities in relativity, measured with rapidity, is a hyperbolic space.

The thinking skill

Find the right parametrisation

Describe your objects by a parameter in which the transformation becomes simple.

Describing directions by their gradient, T acts by a messy formula; describing them by v with gradient $\tanh v$, it simply adds u . The same idea made rotations simple with angles, and made the map $x \mapsto x^2 - 2$ simple in Section 2.8.

Try this next

- 1 Composing boosts.** Show that $T(u)T(w) = T(u + w)$, where $T(u)$ is the matrix in the question. What is $T(u)^{-1}$?
- 2 Time dilation.** A spaceship travels at 0.8 times the speed of light. Find its rapidity u and $\cosh u$. If 10 years pass on Earth, how much time passes on the ship? (Use $t_{\text{ship}} = t_{\text{Earth}} / \cosh u$.)
- 3 Hyperbolic area.** Show that the area enclosed by the hyperbola $x^2 - y^2 = 1$, the x -axis and the line from the origin to $(\cosh v, \sinh v)$ is $\frac{v}{2}$, just as the corresponding area for a circle is $\frac{\theta}{2}$. Why must T preserve such areas?

- 4 Squeeze.** Change coordinates to $p = x + y$, $q = x - y$. Show that T becomes $p \mapsto e^u p$, $q \mapsto e^{-u} q$, and that the hyperbola becomes $pq = 1$. Describe the transformation geometrically in these coordinates.

Further reading

- **Edwin Taylor & John Wheeler, *Spacetime Physics*** (W. H. Freeman). Special relativity with rapidity and spacetime diagrams, for students with A-level mathematics.
- **Tevian Dray, *The Geometry of Special Relativity*** (CRC Press). Relativity as hyperbolic trigonometry, in exactly the spirit of this section.
- **Albert Einstein, *Relativity: The Special and the General Theory*** (various editions). Einstein's own popular account, still very readable.

7.7 Quantities That Never Change

The question

In this question, $\mathbf{M}$ and $\mathbf{N}$ are non-singular 2×2 matrices.

The *trace* of the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $\text{tr}(\mathbf{M}) = a + d$.

- (i) Prove that, for any two matrices $\mathbf{M}$ and $\mathbf{N}$, $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ and derive an expression for $\text{tr}(\mathbf{M} + \mathbf{N})$ in terms of $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{N})$.

The entries in matrix $\mathbf{M}$ are functions of t and $\frac{d\mathbf{M}}{dt}$ denotes the matrix whose entries are the derivatives of the corresponding entries in $\mathbf{M}$.

- (ii) Show that

$$\frac{1}{\det \mathbf{M}} \frac{d}{dt} (\det \mathbf{M}) = \text{tr} \left(\mathbf{M}^{-1} \frac{d\mathbf{M}}{dt} \right).$$

- (iii) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{d\mathbf{M}}{dt} = \mathbf{MN} - \mathbf{NM},$$

where the entries in matrix $\mathbf{N}$ are also functions of t .

Show that $\det \mathbf{M}$, $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{M}^2)$ are independent of t .

In the case $\mathbf{N} = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$, and given that $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ when $t = 0$, find $\mathbf{M}$ as a function of t .

- (iv) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{d\mathbf{M}}{dt} = \mathbf{MN},$$

where the entries in matrix $\mathbf{N}$ are again functions of t .

The trace of $\mathbf{M}$ is non-zero and independent of t . Is it necessarily true that $\text{tr}(\mathbf{N}) = 0$?

The hook

Some of the deepest facts in physics are *conservation laws*: energy, momentum and angular momentum stay the same however complicated the motion. When a system

has enough conserved quantities, its motion can be solved exactly; when it has too few, it can be chaotic.

This question shows a remarkable way of manufacturing conserved quantities. If a matrix changes in time according to $\frac{d\mathbf{M}}{dt} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$, then however wildly its entries vary, its trace and determinant, and hence its eigenvalues, never change. Such a pair $(\mathbf{M}, \mathbf{N})$ is called a *Lax pair*, after Peter Lax, who introduced the idea in 1968 to explain the astonishing behaviour of *solitons*, solitary waves that pass through each other unchanged.

Before you start



Ask yourself The expression $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ is called the *commutator* of $\mathbf{M}$ and $\mathbf{N}$. When is it zero? What is its trace, given part (i)?

It is zero exactly when $\mathbf{M}$ and $\mathbf{N}$ commute. And by part (i) its trace is $\text{tr}(\mathbf{M}\mathbf{N}) - \text{tr}(\mathbf{N}\mathbf{M}) = 0$, always. That single fact is the beginning of every conservation law in the question.

Working it through

STAGE 1 The trace

With $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, the diagonal entries of $\mathbf{M}\mathbf{N}$ are $ap + br$ and $cq + ds$, so $\text{tr}(\mathbf{M}\mathbf{N}) = ap + br + cq + ds$. Swapping the roles of $\mathbf{M}$ and $\mathbf{N}$ gives $pa + qc + rb + sd$, the same. Clearly $\text{tr}(\mathbf{M} + \mathbf{N}) = \text{tr } \mathbf{M} + \text{tr } \mathbf{N}$. ■

STAGE 2 The derivative of a determinant



Ask yourself Differentiate $\det \mathbf{M} = ad - bc$ by the product rule. Then compute

$\mathbf{M}^{-1} \frac{d\mathbf{M}}{dt}$ and its trace, using $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Writing dots for derivatives, $\frac{d}{dt} \det \mathbf{M} = \dot{a}d + a\dot{d} - \dot{b}c - b\dot{c}$. And

$$\mathbf{M}^{-1} \dot{\mathbf{M}} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix}$$

has diagonal entries $\frac{d\dot{a}-b\dot{c}}{\det \mathbf{M}}$ and $\frac{-c\dot{b}+a\dot{d}}{\det \mathbf{M}}$. Their sum is $\frac{1}{\det \mathbf{M}} \frac{d}{dt} \det \mathbf{M}$. ■

STAGE 3 Three conserved quantities

Ask yourself Take the trace of $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$. For the determinant, use part (ii): what is $\mathbf{M}^{-1}(\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M})$?

Trace: $\frac{d}{dt} \operatorname{tr} \mathbf{M} = \operatorname{tr} \dot{\mathbf{M}} = \operatorname{tr}(\mathbf{M}\mathbf{N}) - \operatorname{tr}(\mathbf{N}\mathbf{M}) = 0$.

Determinant: by part (ii),

$$\frac{1}{\det \mathbf{M}} \frac{d}{dt} \det \mathbf{M} = \operatorname{tr}(\mathbf{M}^{-1}(\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M})) = \operatorname{tr} \mathbf{N} - \operatorname{tr}(\mathbf{M}^{-1}\mathbf{N}\mathbf{M}) = \operatorname{tr} \mathbf{N} - \operatorname{tr} \mathbf{N} = 0,$$

using part (i) in the form $\operatorname{tr}(\mathbf{M}^{-1}(\mathbf{N}\mathbf{M})) = \operatorname{tr}((\mathbf{N}\mathbf{M})\mathbf{M}^{-1}) = \operatorname{tr} \mathbf{N}$.

Trace of the square: $\operatorname{tr}(\mathbf{M}^2) = (\operatorname{tr} \mathbf{M})^2 - 2 \det \mathbf{M}$ (see Section 7.4), which is constant because both terms are. ■

STAGE 4 An explicit solution

Pause and try With $\mathbf{N} = t\mathbf{I} + t\mathbf{E}$, where $\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, compute $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ and write down the differential equations for a, b, c, d . Solve them in the order c , then a and d , then b .

The part $t\mathbf{I}$ commutes with everything, so $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M} = t(\mathbf{M}\mathbf{E} - \mathbf{E}\mathbf{M})$. Now $\mathbf{M}\mathbf{E} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$ and $\mathbf{E}\mathbf{M} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$, so

$$\dot{a} = -tc, \quad \dot{b} = t(a - d), \quad \dot{c} = 0, \quad \dot{d} = tc.$$

So $c = C$ is constant; then $a = A - \frac{1}{2}Ct^2$ and $d = D + \frac{1}{2}Ct^2$; then $\dot{b} = t(A - D - Ct^2)$, so $b = B + \frac{1}{2}(A - D)t^2 - \frac{1}{4}Ct^4$. Therefore

$$\mathbf{M} = \begin{pmatrix} A - \frac{1}{2}Ct^2 & B + \frac{1}{2}(A - D)t^2 - \frac{1}{4}Ct^4 \\ C & D + \frac{1}{2}Ct^2 \end{pmatrix}. \quad \blacksquare$$

Check: the trace is $A + D$, and the determinant is $(A - \frac{1}{2}Ct^2)(D + \frac{1}{2}Ct^2) - C(B + \frac{1}{2}(A - D)t^2 - \frac{1}{4}Ct^4) = AD - BC$, constant as promised (Figure 7.7.1).

STAGE 5 A different equation

Ask yourself If $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N}$, then $\mathbf{N} = \mathbf{M}^{-1}\dot{\mathbf{M}}$, and by part (ii) $\operatorname{tr} \mathbf{N}$ is the logarithmic derivative of $\det \mathbf{M}$. Can the trace of $\mathbf{M}$ stay constant while its determinant changes?

Certainly. Take $\mathbf{M} = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ for $|t| < 1$ (where $\det \mathbf{M} = 1 - t^2 \neq 0$). Its trace is constantly 2. Define $\mathbf{N} = \mathbf{M}^{-1}\dot{\mathbf{M}}$, so that $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N}$:

$$\mathbf{N} = \frac{1}{1 - t^2} \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{1 - t^2} \begin{pmatrix} -t & 1 \\ 1 & -t \end{pmatrix},$$

which is non-singular, with $\text{tr } \mathbf{N} = \frac{-2t}{1-t^2} \neq 0$ for $t \neq 0$. So **no**, $\text{tr } \mathbf{N}$ need not be zero. ■

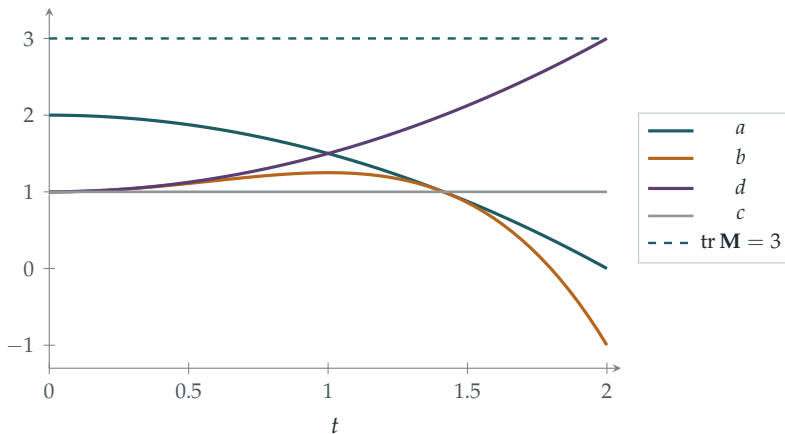


Figure 7.7.1 The entries of $\mathbf{M}(t)$ from Stage 4, with $A = 2$, $B = 1$, $C = 1$, $D = 1$. All four entries except c change, but the trace stays at 3 and the determinant stays at 1 throughout. So the eigenvalues, $\frac{3 \pm \sqrt{5}}{2}$, never change: the flow is *isospectral*.

Looking back

- **The trace of a commutator is zero.** From $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$, every conservation law followed.
- **Logarithmic derivatives.** Part (ii) turned the derivative of a determinant into a trace, which is linear and easy to handle.
- **Check conservation in the explicit solution.** The formula for $\mathbf{M}(t)$ was checked by confirming that its trace and determinant are constant.

What's really going on

Solitons and Lax pairs

In 1834 the Scottish engineer John Scott Russell followed, on horseback, a solitary wave travelling along the Union Canal near Edinburgh for over a mile without losing its shape. Such waves are described by the *Korteweg–de Vries (KdV) equation*, and in the 1960s computer experiments by Norman Zabusky and Martin Kruskal showed that two of these “solitons” can collide and emerge unchanged, as if they were particles. Explaining this needed infinitely many conserved quantities. Peter Lax showed in 1968 that the KdV equation can be written as $\dot{L} = LP - PL$ for a pair of differential operators L and P ; then, exactly as in this question, the eigenvalues of L are conserved, and they provide the infinitely many conservation laws. Systems with a Lax pair are

called *integrable*, and they are among the few nonlinear systems that can be solved exactly.

Isospectral flows

Since the trace and determinant of $\mathbf{M}$ determine its characteristic polynomial $\lambda^2 - (\text{tr } \mathbf{M})\lambda + \det \mathbf{M}$, the eigenvalues of $\mathbf{M}$ do not change with time. The flow $\dot{\mathbf{M}} = [\mathbf{M}, \mathbf{N}]$ moves $\mathbf{M}$ around inside the set of matrices with fixed eigenvalues. In fact, the solution is always of the form $\mathbf{M}(t) = \mathbf{S}(t)^{-1}\mathbf{M}(0)\mathbf{S}(t)$ for some invertible $\mathbf{S}(t)$: the matrix stays similar to its starting value. The same idea runs through physics: in quantum mechanics, an observable $\mathbf{A}$ evolves by $\frac{d\mathbf{A}}{dt} = \frac{i}{\hbar}[\mathbf{H}, \mathbf{A}]$, where $\mathbf{H}$ is the energy, so anything that commutes with the energy is conserved.

Jacobi's formula

Part (ii), $\frac{d}{dt} \ln \det \mathbf{M} = \text{tr}(\mathbf{M}^{-1}\dot{\mathbf{M}})$, holds for matrices of any size and is called *Jacobi's formula*. A consequence: if $\dot{\mathbf{M}} = \mathbf{A}\mathbf{M}$ for a constant matrix $\mathbf{A}$, then $\det \mathbf{M}(t) = e^{t \text{tr } \mathbf{A}} \det \mathbf{M}(0)$. In the study of differential equations this tells you how volumes in phase space grow or shrink; for the equations of classical mechanics the trace is zero and volume is preserved, a result known as *Liouville's theorem*.

The thinking skill

Find what does not change

When something evolves, look for quantities built from it that stay constant. They constrain the motion and often reveal its structure.

Here the trace and determinant were conserved because the rate of change was a commutator. In mechanics, conserved quantities come from symmetries (Noether's theorem), and finding them is often the first step in solving a problem.

Try this next

- 1 Check the solution.** Verify by differentiating that the matrix $\mathbf{M}(t)$ in Stage 4 satisfies $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$.
- 2 Similarity.** Show that if $\mathbf{M}(t) = \mathbf{S}^{-1}\mathbf{M}_0\mathbf{S}$, where $\mathbf{M}_0$ is constant and $\mathbf{S}(t)$ satisfies $\dot{\mathbf{S}} = \mathbf{S}\mathbf{N}$, then $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$. (Hint: $\frac{d}{dt}\mathbf{S}^{-1} = -\mathbf{S}^{-1}\dot{\mathbf{S}}\mathbf{S}^{-1}$.)
- 3 Volume growth.** Let $\mathbf{M}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$. Find $\mathbf{N}$ with $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N}$, and check that $\frac{d}{dt} \ln \det \mathbf{M} = \text{tr } \mathbf{N}$.

- 4 Higher powers.** Show directly that $\frac{d}{dt} \operatorname{tr}(\mathbf{M}^k) = 0$ for every positive integer k when $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$.

Further reading

- **P. G. Drazin & R. S. Johnson, *Solitons: An Introduction*** (Cambridge University Press). The standard undergraduate introduction, including Lax pairs.
- **Thierry Dauxois & Michel Peyrard, *Physics of Solitons*** (Cambridge University Press). Includes the story of Scott Russell's wave.
- **Dwight Neuenschwander, *Emmy Noether's Wonderful Theorem*** (Johns Hopkins University Press). How symmetries give conservation laws.

7.8 Four Squares

The question

Let a, b, c, d, p, q, r and s be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where z_1, z_2, z_3 and z_4 are suitably chosen complex numbers, find expressions L_1, L_2, L_3 and L_4 , each of which is linear in a, b, c and d and also linear in p, q, r and s , such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

The hook

7 is not a sum of two squares, or of three. But it is a sum of four: $7 = 4 + 1 + 1 + 1$. In 1770 Joseph-Louis Lagrange proved that *every* positive whole number is a sum of four squares (allowing 0^2). The theorem had been conjectured by Bachet in 1621 and claimed by Fermat, who as usual left no proof; Euler worked on it for forty years without quite finishing.

The key tool, found by Euler in 1748, is in this question: a product of two sums of four squares is itself a sum of four squares. It means you only need to prove Lagrange's theorem for prime numbers, and the rest follows by multiplying. The question derives the identity in a few lines from determinants of complex matrices, which are the quaternions of Section 6.7 in disguise.

Before you start



Ask yourself Recall the two-square version: $(a^2 + b^2)(p^2 + q^2)$ is a sum of two squares because $|z|^2|w|^2 = |zw|^2$ for complex numbers $z = a + bi$, $w = p + qi$. How could matrices extend this to four squares?

The idea is to find a kind of “number” made from four real numbers whose “size squared” is $a^2 + b^2 + c^2 + d^2$ and which is multiplicative. Matrices are multiplicative under determinants: $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$. So we want a matrix built from a, b, c, d whose determinant is $a^2 + b^2 + c^2 + d^2$, and such that products of such matrices are again of the same form.

Working it through

STAGE 1 The determinant is a sum of four squares

Ask yourself What is the determinant of $\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$? If $z_1 = a + bi$ and $z_2 = c + di$, what is it in terms of a, b, c, d ?

$\det = z_1 z_1^* + z_2 z_2^* = |z_1|^2 + |z_2|^2 = a^2 + b^2 + c^2 + d^2$. Similarly with $z_3 = p + qi$ and $z_4 = r + si$, the second matrix has determinant $p^2 + q^2 + r^2 + s^2$.

STAGE 2 The product has the same form

Multiplying,

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix} = \begin{pmatrix} z_1 z_3 - z_2 z_4^* & z_1 z_4 + z_2 z_3^* \\ -z_2^* z_3 - z_1^* z_4^* & -z_2^* z_4 + z_1^* z_3^* \end{pmatrix}.$$

Writing $w_1 = z_1 z_3 - z_2 z_4^*$ and $w_2 = z_1 z_4 + z_2 z_3^*$, the bottom row is $(-w_2^*, w_1^*)$, so the product is $\begin{pmatrix} w_1 & w_2 \\ -w_2^* & w_1^* \end{pmatrix}$, of the same form. Its determinant is $|w_1|^2 + |w_2|^2$, and since determinants multiply,

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = |w_1|^2 + |w_2|^2.$$

STAGE 3 Reading off the four squares

Pause and try Expand $w_1 = (a + bi)(p + qi) - (c + di)(r - si)$ and $w_2 = (a + bi)(r + si) + (c + di)(p - qi)$ into real and imaginary parts.

$$w_1 = (ap - bq - cr - ds) + i(aq + bp + cs - dr),$$

$$w_2 = (ar - bs + cp + dq) + i(as + br - cq + dp).$$

So

$$L_1 = ap - bq - cr - ds,$$

$$L_2 = aq + bp + cs - dr,$$

$$L_3 = ar - bs + cp + dq,$$

$$L_4 = as + br - cq + dp,$$

each linear in a, b, c, d and in p, q, r, s , with $(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2$. ■

For example, $7 = 2^2 + 1^2 + 1^2 + 1^2$ and $3 = 1^2 + 1^2 + 1^2 + 0^2$. With $(a, b, c, d) = (2, 1, 1, 1)$ and $(p, q, r, s) = (1, 1, 1, 0)$ the formulas give $(L_1, L_2, L_3, L_4) = (0, 2, 4, 1)$, and indeed $0^2 + 2^2 + 4^2 + 1^2 = 21 = 7 \times 3$.

Looking back

- **Multiplicativity does the work.** The identity came from $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$, exactly as the two-square identity comes from $|zw| = |z||w|$.
- **Closure under multiplication.** The special form $\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$ is preserved by products, which is what makes the result another sum of four squares.
- **Bookkeeping matters.** The final step is routine but error-prone; signs and conjugates must be handled carefully.

What's really going on

Lagrange's four-square theorem

Here is the shape of Lagrange's proof. By Euler's identity, it is enough to show every prime p is a sum of four squares. A counting argument shows that some multiple mp , with $m < p$, is a sum of four squares. A clever "descent" then shows that if mp is a sum of four squares with $1 < m < p$, so is $m'p$ for some smaller m' ; repeating, we reach $m = 1$. (Compare Fermat's descent in Section 1.5.) In 1834 Jacobi went much further and found the exact number of ways to write n as a sum of four squares: 8 times the sum of the divisors of n that are not multiples of 4. For example, 1 can be written in 8 ways (± 1 in any of four positions).

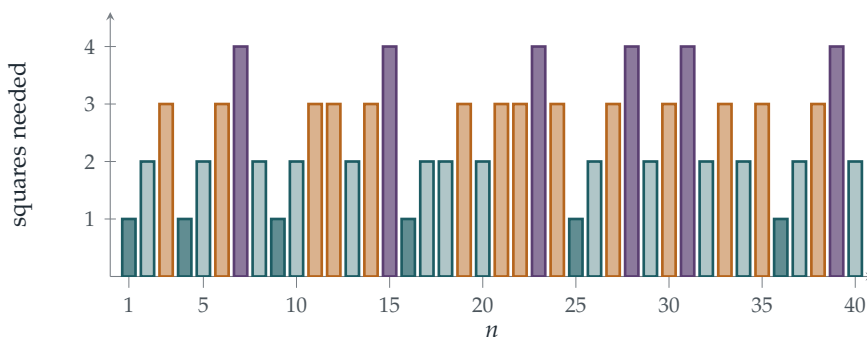


Figure 7.8.1 The least number of squares needed to make each n from 1 to 40. Every number needs at most four, by Lagrange's theorem. The numbers that need four (plum) are 7, 15, 23, 28, 31, 39, $\dots$: exactly those of the form $4^k(8m+7)$, by a theorem of Legendre and Gauss.

Two, three, four

The pattern across dimensions is subtle. Sums of two squares come from complex numbers, and a number is a sum of two squares exactly when its primes of the form $4k+3$ appear to even powers (Section 1.6). Sums of four squares come from quaternions,

and every number is one. But sums of three squares have no multiplication identity at all (there is no three-dimensional number system with a multiplicative size; that is why Hamilton failed with triples, Section 6.7), and the theorem is harder: Legendre and Gauss showed that n is a sum of three squares unless $n = 4^k(8m + 7)$ (Figure 7.8.1). Beyond four, the eight-dimensional octonions give an eight-square identity, found by Degen and Graves, and Hurwitz proved in 1898 that identities of this kind exist only for 1, 2, 4 and 8 squares.

Waring's problem

Lagrange's theorem is the first case of a question asked by Edward Waring in 1770: is there, for each k , a number $g(k)$ such that every positive integer is a sum of at most $g(k)$ k th powers? David Hilbert proved in 1909 that there is. We now know $g(2) = 4$, $g(3) = 9$ (for example $23 = 2 \cdot 2^3 + 7 \cdot 1^3$ needs nine cubes) and $g(4) = 19$.

The thinking skill

Find a multiplicative structure

To show a property is preserved under multiplication, find an algebraic structure (complex numbers, quaternions, matrices) in which the property is the "size" of an object and size is multiplicative.

Once such a structure is found, a product identity comes for free, and a problem about all numbers reduces to a problem about primes.

Try this next

- 1 A worked example.** Use the formulas for $L_1, \dots, L_4$ with $(a, b, c, d) = (p, q, r, s) = (2, 1, 1, 1)$ to write $49 = 7 \times 7$ as a sum of four squares, and check your answer. Is it the same as the obvious $49 = 7^2 + 0^2 + 0^2 + 0^2$?
- 2 Two squares inside four.** Show that if $c = d = r = s = 0$, the identity reduces to the two-square identity of Section 1.6.
- 3 Numbers needing four squares.** Show that no number of the form $8m + 7$ is a sum of three squares, by considering squares modulo 8.
- 4 Jacobi's formula.** Count the representations of 2 as a sum of four squares (with order and signs mattering), and check that the answer is $8 \times (1 + 2) = 24$.

Further reading

- **Harold Davenport**, *The Higher Arithmetic* (Cambridge University Press). Chapter V

proves Lagrange's four-square theorem.

- ▶ **John Conway & Derek Smith, *On Quaternions and Octonions*** (A. K. Peters). The arithmetic of quaternions, including a proof of the four-square theorem via quaternion integers.
- ▶ **Aigner & Ziegler, *Proofs from THE BOOK***, chapter "Representing numbers as sums of two squares". The two-square companion to this section.

7.9 Only Five Platonic Solids

The question

A polyhedron is a solid bounded by F plane faces, which meet in E edges and V vertices. You may assume *Euler's formula*, that $V - E + F = 2$.

In a regular polyhedron the faces are equal regular m -sided polygons, n of which meet at each vertex. Show that

$$F = \frac{4n}{h},$$

where $h = 4 - (n - 2)(m - 2)$.

By considering the possible values of h , or otherwise, prove that there are only five regular polyhedra, and find V , E and F for each.

The hook

There are infinitely many regular polygons: triangles, squares, pentagons, and so on for every number of sides. You might expect infinitely many regular polyhedra too. There are exactly five: the tetrahedron, cube, octahedron, dodecahedron and icosahedron. Plato associated four of them with the elements (fire, earth, air, water) and the fifth with the cosmos; Euclid's *Elements*, the most influential mathematics book ever written, ends with a proof that there are no others; Kepler built a model of the solar system from them.

This question proves the classification in a few lines, from a single remarkable formula found by Euler in 1750: for any convex polyhedron, $V - E + F = 2$. A formula about counting turns into a theorem about shapes, and the number 2 turns out to be a fingerprint of the sphere.

Before you start



Ask yourself Check Euler's formula on a cube: how many vertices, edges and faces does it have? And on a square-based pyramid?

The cube has $V = 8$, $E = 12$, $F = 6$, and $8 - 12 + 6 = 2$. The square-based pyramid has 5 vertices, 8 edges and 5 faces, and $5 - 8 + 5 = 2$. The formula holds for any polyhedron without holes, however irregular.

Ask yourself In a regular polyhedron, each face has m edges and each edge borders two faces. What equation does that give between E and F ? Similarly, between E and V ?

Counting edge–face incidences in two ways: each of the F faces has m edges, and each edge is counted twice (once for each of its two faces), so $mF = 2E$. Similarly each vertex has n edges and each edge has two ends, so $nV = 2E$. This “count the same thing in two ways” trick is called *double counting*.

Working it through

STAGE 1 A formula for F

From $mF = 2E$ and $nV = 2E$, we have $E = \frac{mF}{2}$ and $V = \frac{mF}{n}$. Substituting into Euler’s formula,

$$\frac{mF}{n} - \frac{mF}{2} + F = 2 \implies F \frac{2m - mn + 2n}{2n} = 2 \implies F = \frac{4n}{2m + 2n - mn}.$$

And $4 - (n - 2)(m - 2) = 4 - (mn - 2m - 2n + 4) = 2m + 2n - mn = h$. So $F = \frac{4n}{h}$. ■

STAGE 2 Which values of m and n are possible?

Ask yourself What is the smallest possible value of m (sides of a face) and of n (faces at a vertex)? And since $F > 0$, what sign must h have?

A polygon needs at least 3 sides, so $m \geq 3$, and at least 3 faces must meet at each vertex of a solid, so $n \geq 3$. Since $F > 0$, we need $h > 0$, that is, $(n - 2)(m - 2) < 4$. With both factors at least 1, the only possibilities for $(m - 2, n - 2)$ are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, 3)$ and $(3, 1)$.

Pause and try For each of the five possibilities, compute h , then F , then $E = \frac{mF}{2}$ and $V = \frac{mF}{n}$. Identify each solid.

m	n	h	F	E	V	solid
3	3	3	4	6	4	tetrahedron
3	4	2	8	12	6	octahedron
4	3	2	6	12	8	cube
3	5	1	20	30	12	icosahedron
5	3	1	12	30	20	dodecahedron

These are the only possibilities, so there are at most five regular polyhedra; and all five exist (Euclid shows how to construct each of them). ■

Looking back

- **Double counting.** Counting edge–face and edge–vertex incidences in two ways gave $mF = 2E$ and $nV = 2E$.
- **An inequality from positivity.** The simple requirement $F > 0$ cut infinitely many candidates down to five.
- **Existence is separate.** The argument shows there are *at most* five; that each actually exists has to be checked by constructing it.

What's really going on

Euler's formula and topology

Euler stated $V - E + F = 2$ in 1750, but his proof had gaps, and the formula has a fascinating history of attempted proofs, counterexamples and repairs. A “picture frame” polyhedron, with a square hole through the middle, has $V - E + F = 0$; a polyhedron made of two cubes joined at a vertex has $V - E + F = 3$. Imre Lakatos turned this history into a famous book, *Proofs and Refutations*, about how mathematical concepts are sharpened by counterexamples. The resolution is that $V - E + F$ depends only on the *shape* of the surface in the loose sense of topology: it is 2 for any surface that can be deformed into a sphere, 0 for a torus (a doughnut), and $2 - 2g$ for a surface with g holes. This number, the *Euler characteristic*, was one of the first ideas of topology, the study of properties unchanged by stretching and bending.

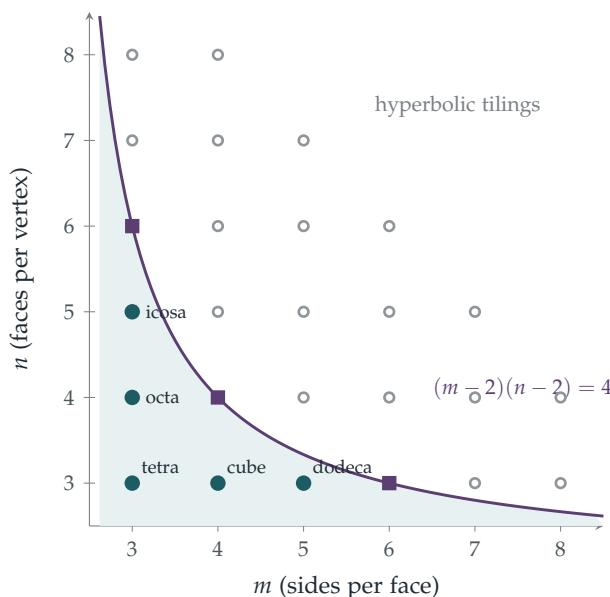


Figure 7.9.1 The possible pairs (m, n) with $m, n \geq 3$. Below the curve $(m-2)(n-2) = 4$ (shaded) are the five Platonic solids (teal). On the curve are three tilings of the flat plane (plum squares): by triangles ($m = 3, n = 6$), squares ($4, 4$) and hexagons ($6, 3$). Every point above the curve (open circles) gives a regular tiling of the hyperbolic plane.

Beyond the sphere

What happens when $h = 0$ or $h < 0$? When $(m-2)(n-2) = 4$, there are three solutions, $(3, 6)$, $(4, 4)$ and $(6, 3)$: six triangles, four squares or three hexagons meet at a point with angles adding up to exactly 360° . These are the three regular tilings of the flat plane, the patterns of bathroom floors and honeycombs. And when $(m-2)(n-2) > 4$, the angles at a vertex add up to *more* than 360° , which is impossible in the flat plane but perfectly possible in the hyperbolic plane of Section 5.8. There are infinitely many regular tilings of the hyperbolic plane, one for every point above the curve in Figure 7.9.1; Escher's *Circle Limit* woodcuts are drawings of some of them. So the classification of Platonic solids is really the first case of a trichotomy (sphere, plane, hyperbolic plane) that runs through all of geometry.

Symmetry groups

The symmetries of the Platonic solids form groups: 12 rotations for the tetrahedron, 24 for the cube and octahedron (which are dual: swap faces and vertices, $m \leftrightarrow n$), and 60 for the dodecahedron and icosahedron. The icosahedral group of order 60 is the “simple” group that, as we mentioned in Section 6.4, prevents the quintic from being solved by radicals. Felix Klein wrote a whole book, *Lectures on the Icosahedron* (1884), connecting the icosahedron to the solution of quintic equations.

The thinking skill

Double counting

Count the same collection in two different ways and equate the results.

Here the collection was the set of (edge, face) pairs with the edge on the face. Double counting is one of the most powerful tools in combinatorics, and it reappears throughout Chapter 9.

Try this next

- 1 **Duality.** Show that swapping m and n swaps V and F and leaves E unchanged. Which pairs of Platonic solids are dual to each other? Which is self-dual?
- 2 **A football.** A traditional football is made of 12 pentagons and 20 hexagons, with three faces meeting at each vertex. Use double counting and Euler's formula to check that $V - E + F = 2$. Show more generally that any such polyhedron made of pentagons and hexagons, three at each vertex, has exactly 12 pentagons.
- 3 **A torus.** Make a "picture frame" polyhedron (a square block with a square hole) and count V , E and F . Check that $V - E + F = 0$.
- 4 **Angles at a vertex.** The interior angle of a regular m -gon is $180^\circ(1 - \frac{2}{m})$. Show that n of them fit round a vertex of a convex solid (total less than 360°) exactly when $(m - 2)(n - 2) < 4$, giving a second proof of the classification.

Further reading

- **David Richeson, *Euler's Gem*** (Princeton University Press). The history of $V - E + F = 2$ and how it led to topology. Excellent for a sixth-former.
- **Imre Lakatos, *Proofs and Refutations*** (Cambridge University Press). A dialogue about Euler's formula and the nature of proof.
- **H. S. M. Coxeter, *Regular Polytopes*** (Dover). The classic book on regular polyhedra and their higher-dimensional cousins.

What We Have Learned

We began by saying that a matrix is a transformation of space with a life of its own, and asked three questions: how matrices are like numbers and how they differ, what a matrix really does, and what stays the same. Here is what the chapter's nine questions showed.

1. How are matrices like numbers, and how are they different?

Matrices can be added and multiplied, but Section 7.1 showed which familiar rules fail. Multiplication is not commutative, so $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$ need not be $\mathbf{A}^2 - \mathbf{B}^2$; there are zero divisors, so $\mathbf{AB} = \mathbf{0}$ does not force $\mathbf{BA} = \mathbf{0}$; and a quadratic matrix equation can have infinitely many roots. Each failure had a geometric explanation, in terms of what the matrices do to the plane. What survives, crucially, is that determinants multiply, $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$. Section 7.3 showed that this property essentially *characterises* the determinant, and Section 7.8 used it to prove Euler's four-square identity, the key step in Lagrange's theorem that every whole number is a sum of four squares.

2. What does a matrix really do?

Several sections answered this by finding the right point of view.

- ▶ **Eigenvectors** (Section 7.2) are the directions a matrix does not turn. In their coordinates a symmetric matrix is just a stretch, and the image of a circle is an ellipse whose axes and area can be read off.
- ▶ **The determinant** (Section 7.3) is the signed area scale factor: shears preserve area, swaps reverse orientation, singular matrices squash the plane flat.
- ▶ **Trace and determinant** (Section 7.4) decide, via the Cayley–Hamilton theorem, whether a matrix returns to the identity after a few steps. A matrix with $\mathbf{M}^8 = \mathbf{I}$ need not be a rotation; it is a rotation seen in skewed coordinates. And because traces of integer matrices are integers, crystals cannot have five-fold symmetry.
- ▶ **Rotations in three dimensions** (Section 7.5) are built from the cross product by Rodrigues' formula, the three-dimensional analogue of $e^{i\theta} = \cos \theta + i \sin \theta$.
- ▶ **Hyperbolic rotations** (Section 7.6) move points along a hyperbola and add their hyperbolic angles. They are the Lorentz boosts of special relativity, and their invariant lines are the paths of light.

3. What stays the same?

Invariants were everywhere. Rotations preserve $x^2 + y^2$, Lorentz boosts preserve $x^2 - y^2$, and similar matrices share their trace and determinant. In Section 7.7, a matrix evolving by $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ kept its trace and determinant, and so its eigenvalues, constant, however much its entries changed: the mechanism behind the conservation laws of solitons. And in Section 7.9, the invariant was a number, $V - E + F = 2$, which holds for every convex polyhedron and pins the regular ones down to just five. That number, the Euler characteristic, was one of the first invariants of topology.

4. A toolkit for linear algebra

Method	Where you used it
Geometric counterexamples	Rules that fail (7.1)
Eigenvectors as natural axes	The ellipse (7.2); boosts (7.6)
Probing with special matrices	The determinant (7.3)
Cayley–Hamilton	Finite order (7.4)
Splitting into parallel and perpendicular parts	Rodrigues’ formula (7.5)
Choosing the right parameter	Rapidity (7.6)
$\text{tr}(\mathbf{M}\mathbf{N}) = \text{tr}(\mathbf{N}\mathbf{M})$	Conserved quantities (7.7)
Multiplicativity of determinants	Four squares (7.8)
Double counting	Platonic solids (7.9)

Where next

- Eigenvalues and eigenvectors govern the vibrations of systems of springs and masses in Section 11.2.
- The eigenvectors of a covariance matrix appear in statistics in Section 13.7.
- Linear differential equations, whose solutions are built from matrix exponentials, are the subject of Chapter 8.

If you want to go further before university

- **3Blue1Brown, *Essence of Linear Algebra***: a free video series that shows what matrices do. Watch it first.
- **Gilbert Strang, *Introduction to Linear Algebra***: the classic first course, with a strong emphasis on understanding and applications.
- **Sheldon Axler, *Linear Algebra Done Right***: a more abstract approach, centred on linear maps; free online.
- **David Richeson, *Euler’s Gem***: Euler’s formula and the birth of topology.

Kaplansky joked that he and Halmos thought without coordinates but computed with

matrices “like fury” behind closed doors. That captures what this chapter has tried to show: matrices are concrete and computable, but behind every calculation there is geometry (a stretch, a rotation, a projection, a boost) and an invariant that does not depend on how you write it down. Learning to move between the two views is one of the most valuable skills a mathematician can have.