

CHAPTER

2

Sequences, Series and Convergence

Infinity is where mathematics gets interesting, and where it is easiest to go wrong.

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Taming the Infinite

The infinite! No other question has ever moved so profoundly the spirit of man.

— David Hilbert

Two and a half thousand years ago, Zeno of Elea argued that motion is impossible. To walk across a room you must first walk half way, then half of what remains, then half of that, and so on for ever: infinitely many tasks, which surely cannot be completed in a finite time. Everyone knew Zeno was wrong, since people do cross rooms, but for a very long time nobody could say exactly *why* he was wrong. The answer is that the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has a finite value, 1. But what does it even *mean* for infinitely many numbers to add up to something?

That question sat at the heart of mathematics for two thousand years. Newton and Leibniz built calculus on top of it in the seventeenth century without ever quite answering it, and their followers produced wonderful results alongside some spectacular nonsense. Euler, the greatest calculator of all time, happily wrote $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$. Others “proved” contradictory things about the same infinite series. It took until the nineteenth century, and the work of Cauchy, Weierstrass and others, before mathematicians had definitions precise enough to tell the true results from the false ones. Those definitions are the subject of this chapter.

Why it matters

Almost every number you will ever compute with is the limit of something. Your calculator does not know $\sqrt{2}$ or π or e : it knows recipes that get closer and closer to them, and it stops when it is close enough. Understanding *how fast* a recipe gets close is the difference between an algorithm that runs in a millisecond and one that never finishes. The same ideas underlie the computer simulations used to forecast weather, design aircraft and price financial products. And on the purest side, the study of limits leads to some of the most beautiful formulas in mathematics: in this chapter you will prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

a result that made Euler famous and that still seems almost magical.

What you will meet at university

At almost every university, *Analysis* is one of the very first courses you take, and for many students it is the biggest shock. In school mathematics you learn to calculate

limits; in Analysis you learn to *prove* things about them, starting from precise definitions. You will define what it means for a sequence to converge, using the phrases “for every $\varepsilon > 0$ ” and “there exists N ” that appear in Section 2.1. You will prove that every bounded increasing sequence converges, and discover that this depends on a property of the real numbers, *completeness*, that the rational numbers do not have. You will study tests for the convergence of series, and then use the same ideas to give rigorous meaning to continuity, differentiation and integration.

Later courses build on this foundation in many directions. *Numerical analysis* asks how quickly approximations converge and how to make them faster (Section 2.7, Section 2.6). *Dynamical systems* studies what happens when you apply a rule over and over, sometimes reaching a limit and sometimes producing chaos (Section 2.2, Section 2.8). *Complex analysis* explains why π appears in so many infinite sums and products (Section 2.4, Section 2.5). And *analytic number theory* uses limits to answer questions about whole numbers, as in Section 2.9.

The questions that run through this chapter

Three big questions

1. **What does it mean for an infinite process to have an answer?** How do we say precisely that a sequence “gets close to” a limit, or that an infinite sum “adds up to” a number?
2. **How can we prove that something converges without knowing what it converges to?** And once we know a limit exists, how do we find it?
3. **How fast?** Two processes can have the same limit but approach it at wildly different speeds. Why, and how do we measure it?

The first section (Section 2.1) is about language: the precise way mathematicians talk about “eventually”. The next five sections are about famous limits: the golden ratio as a continued fraction, a divergent series with a convergent cousin, $\pi^2/6$, Viète’s product for $2/\pi$, and Gauss’s arithmetic–geometric mean. The last three are about speed and structure: linear versus quadratic convergence, an iteration that either explodes or wanders chaotically, and a family of integrals that close in on e .

Tools for convergence

You will meet a small number of tools again and again, each in a different setting.

§	Topic	Tool
2.1	“Eventually smaller”	Quantifiers: producing m , negating statements
2.2	The golden ratio	Monotone bounded subsequences
2.3	Harmonic and Kempner series	Comparison by blocks
2.4	$\pi^2/6$	Roots of a polynomial, then a squeeze
2.5	Viète’s product	Telescoping, then one simple limit
2.6	The AGM	Two sequences squeezing together; an invariant
2.7	Approximating $\sqrt{2}$	Error recurrences: linear and quadratic
2.8	$x \mapsto x^2 - 2$	A substitution that makes the map simple
2.9	Fractions for e	Integrals that are small but not zero

How to read this chapter

As in Chapter 1, each section starts with a STEP question exactly as set. Try it before reading on. This chapter’s questions often tell you which tool to use (“by considering...”, “you may use the theorem...”); as you work through them, notice how often the same few tools reappear, because at university you will have to choose them yourself.

Several of the questions ask you to assume that limits exist, or that infinite sums and products converge. Where that happens, the section explains what would be needed to prove it. Being aware of the gap between “it obviously works” and “I can prove it works” is the first step towards thinking like an analyst.

2.1 What "Eventually" Means

The question

If $s_1, s_2, s_3, \dots$ and $t_1, t_2, t_3, \dots$ are sequences of positive numbers, we write

$$(s_n) \leq (t_n)$$

to mean

“there exists a positive integer m such that $s_n \leq t_n$ whenever $n \geq m$ ”.

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m ; in the case of a false statement, you should give a counterexample.

- (i) $(1000n) \leq (n^2)$.
- (ii) If it is not the case that $(s_n) \leq (t_n)$, then it is the case that $(t_n) \leq (s_n)$.
- (iii) If $(s_n) \leq (t_n)$ and $(t_n) \leq (u_n)$, then $(s_n) \leq (u_n)$.
- (iv) $(n^2) \leq (2^n)$.

The hook

Almost every statement in a first university analysis course has the same shape: *there exists* something such that *for all* somethings, a condition holds. “The sequence tends to L ” means: for every tolerance $\varepsilon > 0$ *there exists* a point N such that *for all* $n \geq N$, $|a_n - L| < \varepsilon$. Students who are fluent with this shape find analysis exciting. Students who aren’t find it baffling.

This question is a gentle, beautifully designed introduction to that way of speaking. It defines one new idea, “eventually smaller than”, and asks four true-or-false questions about it. One of the four is false, and the reason it is false is exactly the trap that catches most first-year students when they first try to *negate* a statement with quantifiers in it.

Before you start

Read the definition slowly. $(s_n) \leq (t_n)$ does not say that $s_n \leq t_n$ for every n . It says that *from some point onwards* it is true. The first few terms don’t matter at all: you can change

a million terms at the start of either sequence without affecting whether $(s_n) \leq (t_n)$.



Ask yourself The definition contains the phrase “there exists a positive integer m ”. To prove that $(s_n) \leq (t_n)$, what exactly do you have to produce? To prove that it is *false*, what do you have to show?

To prove it, you must *produce* a suitable m and then show that *every* $n \geq m$ works. To disprove it, you must show that *no* m works: whatever m someone proposes, there is some $n \geq m$ with $s_n > t_n$. Keep that second version in mind for part (ii).

Working it through

STAGE 1 A linear function against a quadratic

Part (i) asks whether $1000n \leq n^2$ eventually. Since $n > 0$, this is the same as $1000 \leq n$. So $m = 1000$ works: for every $n \geq 1000$, $n^2 = n \cdot n \geq 1000n$. The statement is **true**. ■

It doesn't matter that $1000n$ is far bigger than n^2 for the first 999 terms. “Eventually” ignores any finite number of terms.

STAGE 2 Is “eventually smaller” like ordinary “smaller”?

For ordinary numbers, if it is not the case that $a \leq b$, then $b < a$, so certainly $b \leq a$. Part (ii) asks whether the same is true for our new relation.



Ask yourself What does “it is *not* the case that $(s_n) \leq (t_n)$ ” actually mean? Write it out in full, without the word “not” in front.

The statement “there exists m such that for all $n \geq m$, $s_n \leq t_n$ ” is false precisely when

$$\text{for every } m \text{ there is some } n \geq m \text{ with } s_n > t_n.$$

In other words, $s_n > t_n$ happens *infinitely often*. That is much weaker than “ $s_n > t_n$ from some point on”. The sequences could keep swapping places.



Pause and try Find two sequences of positive numbers that swap places for ever, so that neither is eventually smaller than the other.

Take $s_n = 2 + (-1)^n$, so $s_n = 1, 3, 1, 3, \dots$, and $t_n = 2$ for all n . For every m there is an even $n \geq m$ with $s_n = 3 > 2 = t_n$, so it is not the case that $(s_n) \leq (t_n)$. But equally, for every m there is an odd $n \geq m$ with $t_n = 2 > 1 = s_n$, so it is not the case that $(t_n) \leq (s_n)$ either. The statement is **false**. ■

STAGE 3 Chaining



Ask yourself In part (iii) you are given two values of m , one for each hypothesis. They might be different. Which one should you use for the conclusion?

Suppose $s_n \leq t_n$ for all $n \geq m_1$ and $t_n \leq u_n$ for all $n \geq m_2$. If $n \geq \max(m_1, m_2)$, both inequalities hold, so $s_n \leq t_n \leq u_n$. So $m = \max(m_1, m_2)$ works, and the statement is **true**. ■

Taking the larger of two thresholds is a move you will make hundreds of times in analysis. It is how you prove, for example, that the sum of two convergent sequences converges.

STAGE 4 A quadratic against an exponential

Try a few values first:

n	1	2	3	4	5	6	7	8
n^2	1	4	9	16	25	36	49	64
2^n	2	4	8	16	32	64	128	256

So $n^2 \leq 2^n$ at $n = 1, 2$, fails at $n = 3$, and then seems to hold from $n = 4$ onwards (Figure 2.1.1). The natural guess is $m = 4$.



Ask yourself To prove $n^2 \leq 2^n$ for all $n \geq 4$ by induction, you need to get from $n^2 \leq 2^n$ to $(n+1)^2 \leq 2^{n+1} = 2 \cdot 2^n$. What inequality between $(n+1)^2$ and n^2 would make that step work?

It would be enough to know that $(n+1)^2 \leq 2n^2$, because then $(n+1)^2 \leq 2n^2 \leq 2 \cdot 2^n$. Now $2n^2 - (n+1)^2 = n^2 - 2n - 1 = (n-1)^2 - 2$, which is positive when $n \geq 3$. So:

► *Base case:* $4^2 = 16 = 2^4$.

► *Inductive step:* if $n \geq 4$ and $n^2 \leq 2^n$, then $(n+1)^2 \leq 2n^2 \leq 2^{n+1}$.

Hence $n^2 \leq 2^n$ for all $n \geq 4$, and the statement is **true** with $m = 4$. ■

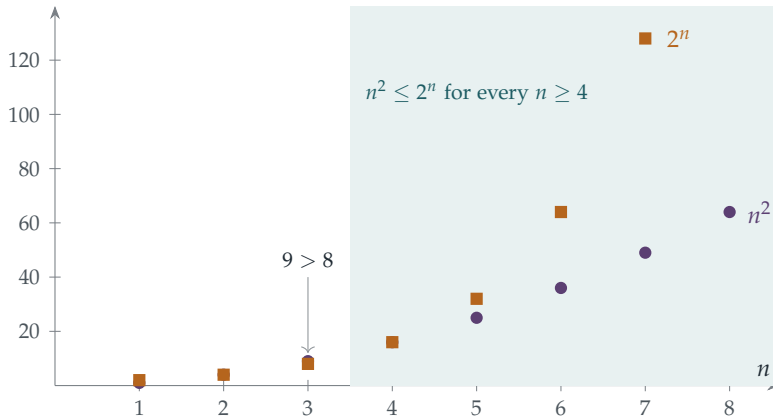


Figure 2.1.1 n^2 (plum circles) against 2^n (amber squares). The two sequences cross twice in the first few terms, but from $n = 4$ onwards 2^n stays ahead for ever. That is exactly what $(n^2) \leq (2^n)$ means, with $m = 4$.

Looking back

- **Proving “there exists” means producing.** In parts (i), (iii) and (iv) the proof consisted of naming m and checking it.
- **Negating swaps the quantifiers.** “Not (there exists m , for all $n \geq m$, P)” becomes “for all m , there exists $n \geq m$, not P ”. Getting this right is what part (ii) tests.
- **A new relation need not behave like an old one.** “Eventually smaller than” is transitive, like $\leq$, but unlike $\leq$ it does not compare every pair of sequences.

What’s really going on

The language of analysis

For about 150 years after Newton and Leibniz, mathematicians used limits and infinitesimals brilliantly but could not say exactly what they meant. Bishop Berkeley mocked infinitesimals in 1734 as “the ghosts of departed quantities”, and he had a point. The repair came in the nineteenth century, when Augustin-Louis Cauchy and then Karl Weierstrass replaced vague talk of “becoming infinitely close” with statements built from “for every” and “there exists”. The modern definition

$$a_n \rightarrow L \iff \text{for every } \varepsilon > 0 \text{ there exists } N \text{ such that } |a_n - L| < \varepsilon \text{ for all } n \geq N$$

has exactly the shape of this question’s definition, with one extra quantifier at the front. It is the first thing you will meet in a university analysis course.

Orders of growth

Part (iv) is one small piece of a famous ladder. For any fixed $k > 0$ and $c > 1$,

$$(\ln n) \leq (n^k), \quad (n^k) \leq (c^n), \quad (c^n) \leq (n!), \quad (n!) \leq (n^n),$$

and in fact each sequence is not just eventually smaller than the next but eventually smaller than *any constant multiple* of it. G.H. Hardy wrote a whole book, *Orders of Infinity* (1910), about this kind of comparison. Today it is the everyday language of computer science, where “this algorithm takes $O(n^2)$ steps” means that the number of steps is eventually at most a constant times n^2 . The difference between n^2 and 2^n is the difference between a program that finishes and one that will not finish before the Sun burns out.

Orders that are not total

Part (ii) shows that “eventually smaller” is not a *total order*: some pairs of sequences cannot be compared at all. Relations like this, transitive but not comparing everything, are called *preorders* or *partial orders*, and they turn up everywhere: divisibility of integers, inclusion of sets, and “is a subgroup of” are all examples. Recognising which properties a relation has, and which it doesn’t, is part of the training of a mathematician.

The thinking skill

Handling quantifiers

To prove “there exists m such that for all $n \geq m \dots$ ”, name m and then check every n . To disprove it, show that for every m some $n \geq m$ fails.

When a statement has several quantifiers, write it out in full before trying to prove or disprove it, and negate it by working from the left: each “for all” becomes “there exists”, each “there exists” becomes “for all”, and the condition at the end is negated.

Try this next

- 1 The specimen warm-up.** (From the STEP III Specimen paper, issued before the first STEP in 1987.) Show by using the binomial expansion that $(1+x)^n > nx$ whenever $x \geq 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K , an integer N can be found such that $y^n > K$ for all $n \geq N$. Show further that if $y > 1$ then, given any K , an integer N can be found such that $y^n/n > K$ for all $n \geq N$. (Hint for the last part: use the next term of the binomial expansion.)
- 2 Both ways round.** Prove that if $(s_n) \leq (t_n)$ and $(t_n) \leq (s_n)$, then $s_n = t_n$ for all sufficiently large n .

- 3 Racing to infinity.** Find two *increasing* sequences of positive numbers, both tending to infinity, neither of which is eventually smaller than the other.
- 4 Up the ladder.** Show that $(2^n) \leq (n!)$, giving an explicit m . Then show that $(n^{10}) \leq (2^n)$ and find the smallest m that works. (A spreadsheet helps for the second part; the answer is surprisingly large.)

Further reading

- **Lara Alcock**, *How to Study for a Mathematics Degree* (Oxford University Press). Chapter 3 is about exactly this kind of logical language and why new undergraduates find it hard.
- **Lara Alcock**, *How to Think About Analysis* (Oxford University Press). A friendly guide to the definitions of first-year analysis.
- **Daniel Velleman**, *How to Prove It* (Cambridge University Press). A systematic introduction to logic, quantifiers and proof.

2.2 The Golden Ratio as a Limit

The question

In this question, the following theorem may be used.

Let $u_1, u_2, \dots$ be a sequence of (real) numbers. If the sequence is bounded above (that is, $u_n \leq b$ for all n , where b is some fixed number) and increasing (that is, $u_n \geq u_{n-1}$ for all n), then the sequence tends to a limit (that is, converges).

The sequence $u_1, u_2, \dots$ is defined by $u_1 = 1$ and

$$u_{n+1} = 1 + \frac{1}{u_n} \quad (n \geq 1). \quad (*)$$

(i) Show that, for $n \geq 3$,

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1 + u_n)(1 + u_{n-2})}.$$

(ii) Prove, by induction or otherwise, that $1 \leq u_n \leq 2$ for all n .

(iii) Show that the sequence $u_1, u_3, u_5, \dots$ tends to a limit, and that the sequence $u_2, u_4, u_6, \dots$ tends to a limit. Find these limits and deduce that the sequence $u_1, u_2, u_3, \dots$ tends to a limit.

Would this conclusion change if the sequence were defined by (*) and $u_1 = 3$?

The hook

Write down the expression

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

and ask what it equals. A quick answer: call it x ; the part under the first fraction line is x again, so $x = 1 + 1/x$, which gives $x^2 = x + 1$ and $x = \frac{1+\sqrt{5}}{2}$, the golden ratio φ .

But is that argument valid? It assumes the infinite expression *means* something, that the process of building it up settles down to a single number. The same trick applied to $1 - 1 + 1 - 1 + \dots$ ("call it S ; then $S = 1 - S$, so $S = \frac{1}{2}$ ") produces nonsense. This question shows how to do the job properly, using the only tool a STEP candidate has

for proving convergence without knowing the limit in advance: *monotone sequences*.

Before you start

Compute some terms. With $u_1 = 1$:

$$1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \frac{13}{8}, \quad \frac{21}{13}, \quad \frac{34}{21}, \quad \dots$$



Ask yourself Do you recognise the numerators and denominators? And look at the decimals: 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, ... Is the sequence increasing? Decreasing?

They are ratios of consecutive Fibonacci numbers. The sequence is neither increasing nor decreasing: it zig-zags, overshooting and undershooting. So the theorem we are given, which is about increasing sequences, cannot be applied to the whole sequence. But the odd-numbered terms 1, 1.5, 1.6, 1.615, ... seem to go up, and the even-numbered terms 2, 1.667, 1.625, ... seem to go down. That is the plan of the question.

Working it through

STAGE 1 Comparing terms two apart

Part (i) compares u_{n+2} with u_n , two steps apart. The natural first move is to use (*) to write each of them in terms of the term before.



Ask yourself Using (*) twice, write $u_{n+2} - u_n$ in terms of $u_{n+1} - u_{n-1}$, and then $u_{n+1} - u_{n-1}$ in terms of $u_n - u_{n-2}$.

From (*),

$$u_{n+2} - u_n = \frac{1}{u_{n+1}} - \frac{1}{u_{n-1}} = -\frac{u_{n+1} - u_{n-1}}{u_{n+1}u_{n-1}},$$

and in the same way $u_{n+1} - u_{n-1} = -\frac{u_n - u_{n-2}}{u_n u_{n-2}}$. Putting these together, the two minus signs cancel:

$$u_{n+2} - u_n = \frac{u_n - u_{n-2}}{u_{n+1}u_n \cdot u_{n-1}u_{n-2}}.$$

Finally, (*) says $u_{n+1}u_n = u_n + 1$ and $u_{n-1}u_{n-2} = u_{n-2} + 1$, which gives the required formula. (We need $n \geq 3$ so that u_{n-2} exists.) ■

STAGE 2 A trap



Pause and try Prove that $1 \leq u_n \leq 2$ for all n .

It is true for $u_1 = 1$. If $1 \leq u_n \leq 2$, then $\frac{1}{2} \leq \frac{1}{u_n} \leq 1$, so $\frac{3}{2} \leq u_{n+1} \leq 2$, which is certainly between 1 and 2. By induction, $1 \leq u_n \leq 2$ for all n . ■

In particular every u_n is positive, so the denominator $(1 + u_n)(1 + u_{n-2})$ in part (i) is positive.

STAGE 3 Two monotone subsequences



Ask yourself The formula in part (i) says that $u_{n+2} - u_n$ is a positive multiple of $u_n - u_{n-2}$. What does that tell you about the signs of $u_3 - u_1$, $u_5 - u_3$, $u_7 - u_5$, ...?

They all have the same sign! Since $u_3 - u_1 = \frac{3}{2} - 1 > 0$, part (i) and induction give $u_5 > u_3$, $u_7 > u_5$, and so on: the odd terms are increasing. They are bounded above by 2, so by the theorem they tend to a limit; call it L .

Similarly $u_4 - u_2 = \frac{5}{3} - 2 < 0$, so the even terms are decreasing. They are bounded *below* by 1. The theorem is stated for increasing sequences, but $-u_2, -u_4, -u_6, \dots$ is increasing and bounded above by -1 , so it converges, and therefore so do the even terms; call their limit M .

STAGE 4 Finding the limits



Ask yourself To find L , you need an equation relating u_{n+2} to u_n directly. Can you get one from (*)?

Applying (*) twice,

$$u_{n+2} = 1 + \frac{1}{u_{n+1}} = 1 + \frac{1}{1 + 1/u_n} = 1 + \frac{u_n}{u_n + 1} = \frac{2u_n + 1}{u_n + 1}.$$

Letting $n \rightarrow \infty$ through odd values, both u_n and u_{n+2} tend to L , so

$$L = \frac{2L + 1}{L + 1} \implies L^2 + L = 2L + 1 \implies L^2 - L - 1 = 0.$$

The roots are $\frac{1 \pm \sqrt{5}}{2}$, and since $L \geq 1$ we must have $L = \frac{1 + \sqrt{5}}{2} = \varphi$. Exactly the same argument through even values gives $M = \varphi$.



Ask yourself The odd terms tend to φ and the even terms tend to φ . Why does this mean that the whole sequence tends to φ ?

Intuitively, every term is either odd-numbered or even-numbered, and both kinds are getting close to φ . Precisely: given any tolerance $\varepsilon > 0$, the odd terms are within ε of φ from some point N_1 onwards, and the even terms from some point N_2 onwards. So *all* terms are within ε of φ from $\max(N_1, N_2)$ onwards, which is what convergence means. (This is the “take the larger threshold” move from Section 2.1.)

STAGE 5 Starting somewhere else

With $u_1 = 3$ the sequence is

$$3, \quad \frac{4}{3}, \quad \frac{7}{4}, \quad \frac{11}{7}, \quad \frac{18}{11}, \quad \frac{29}{18}, \quad \dots$$

Now $u_1 = 3$ is outside $[1, 2]$, so the induction in part (ii) does not start. But $u_2 = \frac{4}{3}$ is inside, and the induction works from there: $1 \leq u_n \leq 2$ for all $n \geq 2$. Part (i) only ever uses terms from u_{n-2} onwards and only needs them to be positive, which they all are. This time $u_3 - u_1 = \frac{7}{4} - 3 < 0$ and $u_4 - u_2 = \frac{11}{7} - \frac{4}{3} > 0$, so the odd terms decrease and the even terms increase: the roles are swapped, but each subsequence is still monotone and bounded. The limits satisfy the same equation, and both are positive, so both are φ . The conclusion does **not** change. ■

Looking back

- **When a sequence isn't monotone, look for monotone pieces.** Splitting into odd and even terms turned a zig-zag into two well-behaved sequences.
- **Prove convergence first, find the limit second.** The equation $L = (2L + 1)/(L + 1)$ is only legitimate once we know L exists. That is exactly the step the quick argument in the hook skipped.
- **Check which root.** The limit equation had two solutions; the bounds from part (ii) told us which one was right.

What's really going on

Cobwebs and fixed points

Every sequence defined by $u_{n+1} = f(u_n)$ can be drawn as a *cobweb diagram* (Figure 2.2.1). Plot $y = f(x)$ and $y = x$. Start at u_1 on the x -axis, go vertically to the curve (height $u_2 = f(u_1)$), then horizontally to the line $y = x$ (which moves you to $x = u_2$), then vertically to the curve again, and so on. Here $f(x) = 1 + 1/x$ is decreasing, which is why the path spirals inwards, alternately overshooting and undershooting, towards the point where the curve crosses the line: the *fixed point* φ , where $f(\varphi) = \varphi$.

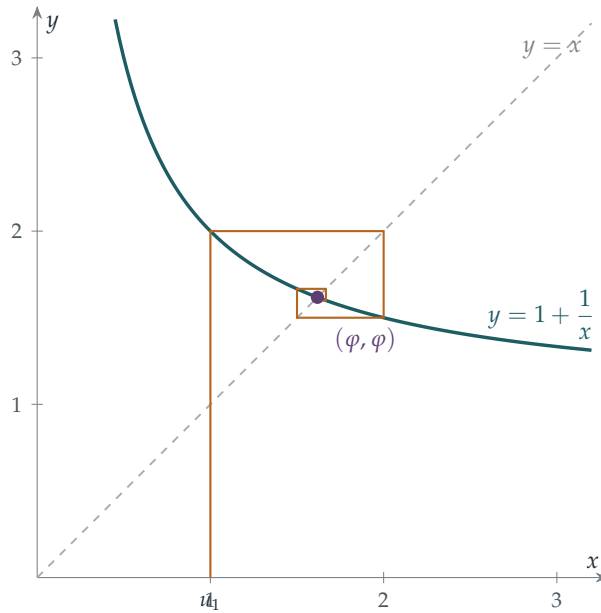


Figure 2.2.1 A cobweb diagram for $u_{n+1} = 1 + 1/u_n$ starting from $u_1 = 1$. The path goes up to the curve and across to the line $y = x$, over and over. Because the curve slopes downwards, the path spirals in towards the fixed point $\phi \approx 1.618$, alternating sides: that is the zig-zag of odd and even terms.

How fast, and why

Where does the factor $(1 + u_n)(1 + u_{n-2})$ in part (i) come from? Near the fixed point, $u_{n+1} - \phi = f(u_n) - f(\phi) \approx f'(\phi)(u_n - \phi)$, and $f'(\phi) = -1/\phi^2 \approx -0.38$. So each step multiplies the error by about -0.38 : the sign flips (zig-zag) and the size shrinks (convergence). Two steps multiply it by $1/\phi^4 \approx 0.146$, which is what $1/((1 + u_n)(1 + u_{n-2}))$ tends to. At university this becomes the *contraction mapping theorem*: if $|f'| \leq k < 1$ near a fixed point, iteration converges to it, with the error shrinking at least geometrically. It is one of the most useful theorems in mathematics, and it is how you prove that differential equations have solutions (Chapter 8). In Section 2.8 we will meet a map whose fixed points are *not* attracting, with very different results.

The most irrational number

The terms $u_n = F_{n+1}/F_n$ are the best possible fractional approximations to ϕ for their size, but “best” is not very good: ϕ is harder to approximate by fractions than any other number. The reason is visible in its continued fraction $[1; 1, 1, 1, \dots]$: large entries in a continued fraction mean very good approximations (think of $\pi = [3; 7, 15, 1, 292, \dots]$ and the excellent $\frac{355}{113}$), and ϕ has the smallest possible entries all the way. Adolf Hurwitz proved in 1891 that every irrational x has infinitely many fractions with $|x - p/q| < 1/(\sqrt{5}q^2)$, and that $\sqrt{5}$ cannot be improved, precisely because of ϕ . This “maximal irrationality” is the reason the golden angle appears in the spiral patterns of

sunflower seeds.

The thinking skill

Monotone and bounded implies convergent

To show a sequence converges without knowing its limit, find a monotone, bounded sequence, or split the sequence into monotone, bounded pieces.

Only once you know a limit exists may you find it by “letting $n \rightarrow \infty$ ” in the recurrence. At university this theorem is a consequence of the *completeness* of the real numbers, the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$, and you will see it proved from the axioms.

Try this next

- 1 **Fibonacci inside.** Prove that $u_n = F_{n+1}/F_n$, where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
- 2 **The error shrinks.** Show that $u_{n+1} - \varphi = -\frac{u_n - \varphi}{\varphi u_n}$, and deduce that $|u_n - \varphi| \leq \varphi^{-(n-1)}$ for all n . (Use $\varphi^2 = \varphi + 1$, and $u_n \geq 1$.)
- 3 **Breaking the sequence.** Show that if $u_1 = \frac{1-\sqrt{5}}{2}$ then the sequence is constant. Find all the starting values u_1 for which some $u_n = 0$, so that the sequence cannot continue. (*Hint:* work backwards from $u_n = 0$.)
- 4 **A cousin for $\sqrt{2}$.** Let $v_1 = 1$ and $v_{n+1} = 1 + \frac{1}{1+v_n}$. Adapt the method of this question to show that $v_n \rightarrow \sqrt{2}$, and write $\sqrt{2}$ as a continued fraction.

Further reading

- **Mario Livio, *The Golden Ratio*** (Broadway). The history of φ , separating the genuine mathematics from the many myths.
- **C. D. Olds, *Continued Fractions*** (Mathematical Association of America). A short, very readable introduction aimed at strong school students.
- **J. C. Burkill, *A First Course in Mathematical Analysis*** (Cambridge University Press). Rigorous treatment of monotone sequences and completeness, with many exercises.

2.3 A Divergent Series and Its Convergent Cousin

The question

The numbers $f(r)$ satisfy $f(r) > f(r+1)$ for $r = 1, 2, \dots$. Show that, for any non-negative integer n ,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n),$$

where k is an integer greater than 1.

(i) By taking $f(r) = 1/r$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $f(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n .

The hook

The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is the most famous example of a sum whose terms shrink to zero but which nevertheless grows without limit. It grows absurdly slowly: you need more than 12,000 terms to pass 10, and about 1.5×10^{43} terms to pass 100. But it does pass every number eventually.

Now throw away every term whose denominator contains the digit 2: out go $\frac{1}{2}, \frac{1}{12}, \frac{1}{20}$ to $\frac{1}{29}, \frac{1}{32}$, and so on. You might expect that removing a tenth or so of the terms would make little difference to a sum that is infinite anyway. Instead, what is left adds up to

less than 80. This surprising fact was proved by A.J. Kempner in 1914, and the question walks you to it with one beautifully simple idea: chop the sum into blocks.

Before you start



Ask yourself The first inequality is about the block of terms from $r = k^n$ to $r = k^{n+1} - 1$. With $k = 2$ and $n = 3$, which terms are these, and how many of them are there?

They are $f(8), f(9), \dots, f(15)$: eight terms. In general the block runs from k^n to $k^{n+1} - 1$, so it has $k^{n+1} - k^n = k^n(k - 1)$ terms. The inequality says the block sum lies between (number of terms) $\times$ (smallest possible term) and (number of terms) $\times$ (largest term).

Working it through

STAGE 1 Bounding a block

Since f is decreasing, every term in the block satisfies

$$f(k^{n+1}) < f(k^{n+1} - 1) \leq f(r) \leq f(k^n) \quad (k^n \leq r \leq k^{n+1} - 1).$$

Adding up the $k^n(k - 1)$ terms gives the inequality. ■

STAGE 2 The harmonic series



Ask yourself With $k = 2$ and $f(r) = 1/r$, what are the two bounds for the block from 2^n to $2^{n+1} - 1$?

The lower bound is $2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$ and the upper bound is $2^n \cdot \frac{1}{2^n} = 1$. So *every* block, however far out, contributes between $\frac{1}{2}$ and 1 (Figure 2.3.1). The blocks for $n = 0, 1, \dots, N$ fit together exactly to make the sum from $r = 1$ to $r = 2^{N+1} - 1$, so adding $N + 1$ blocks gives

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

As $N \rightarrow \infty$ the left-hand side grows without bound, so the partial sums are unbounded and the series diverges. ■

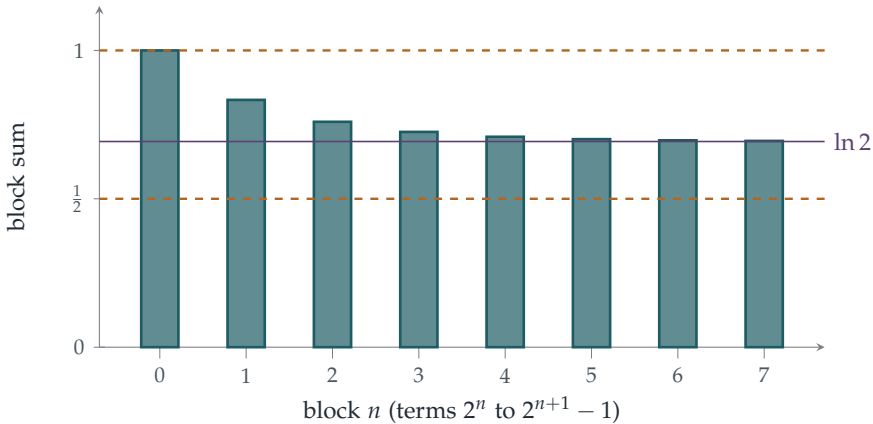


Figure 2.3.1 The harmonic series cut into blocks of 1, 2, 4, 8, ... terms. Every block sum lies between $\frac{1}{2}$ and 1 (amber, dashed), and in fact the block sums settle down towards $\ln 2 \approx 0.693$ (plum). Infinitely many blocks, each worth at least $\frac{1}{2}$, cannot have a finite total.

STAGE 3 The sum of reciprocal cubes

Now take $f(r) = 1/r^3$, still with $k = 2$. The upper bound for block n is

$$2^n \cdot \frac{1}{(2^n)^3} = \frac{1}{4^n}.$$

The blocks for $n = 0, 1, 2, \dots$ cover every positive integer exactly once, so

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} = 1\frac{1}{3}. \quad \blacksquare$$

(Strictly, this shows every partial sum is at most $\frac{4}{3}$; an increasing sequence that is bounded above converges, so the infinite sum exists and is at most $\frac{4}{3}$. The true value is 1.2020...)

STAGE 4 Counting numbers without a 2

Ask yourself How many whole numbers from 0 to 999 can you write using only the digits 0, 1, 3, 4, 5, 6, 7, 8, 9, if you allow leading zeros (so 7 is written 007)?

Each of the three digit positions has 9 choices, so there are 9^3 such strings. They correspond exactly to the numbers from 0 to 999 with no 2, and removing 0 (which is not positive) leaves $9^3 - 1 = 728$ numbers in $S(1000)$. $\blacksquare$

STAGE 5 Kempner's series

The block idea now works with blocks defined by the *number of digits*, which is the same as using $k = 10$.



Pause and try How many d -digit numbers have no 2 in them? And what is the smallest a d -digit number can be?

The first digit can be any of $1, 3, 4, \dots, 9$ (eight choices: not 0, not 2), and each of the remaining $d - 1$ digits has nine choices. So there are $8 \times 9^{d-1}$ such numbers, and each one is at least 10^{d-1} . So the d -digit numbers in $S(n)$ contribute at most

$$8 \times 9^{d-1} \times \frac{1}{10^{d-1}} = 8 \left(\frac{9}{10} \right)^{d-1}$$

to $\sigma(n)$. Adding over all digit lengths $d = 1, 2, 3, \dots$,

$$\sigma(n) \leq \sum_{d=1}^{\infty} 8 \left(\frac{9}{10} \right)^{d-1} = \frac{8}{1 - \frac{9}{10}} = 80.$$

For the strict inequality, notice that most of the bounds are far from sharp (for instance $\frac{1}{3} < 1$, while we have bounded every one-digit term by 1), so $\sigma(n) < 80$ for all n . ■

Looking back

- **Group the terms.** Neither sum is easy term by term. Grouped into blocks of the right size, each was compared with a geometric series.
- **The right blocks depend on the problem.** Powers of 2 suited $1/r$ and $1/r^3$; powers of 10 suited a question about decimal digits.
- **Bounded and increasing.** Both convergence proofs rested on the same principle as Section 2.2: partial sums of positive terms increase, so a bound is all you need.

What's really going on

The condensation test

The first inequality in the question is the heart of *Cauchy's condensation test*: if f is positive and decreasing, then

$$\sum_{r=1}^{\infty} f(r) \text{ converges} \iff \sum_{n=0}^{\infty} 2^n f(2^n) \text{ converges.}$$

It turns hard series into easy ones. Applied to $f(r) = 1/r^p$, it gives $\sum 2^n \cdot 2^{-np} = \sum (2^{1-p})^n$, a geometric series that converges exactly when $p > 1$. Applied to $1/(r \ln r)$ it shows that even this diverges, and to $1/(r(\ln r)^2)$ that this converges. The argument goes back to Nicole Oresme, who proved the harmonic series diverges in about 1350 by exactly the blocks of Figure 2.3.1.

How slowly the harmonic series diverges

Figure 2.3.1 suggests that each block contributes about $\ln 2$, so the sum of the first N terms should be roughly $\log_2 N \times \ln 2 = \ln N$. In fact

$$1 + \frac{1}{2} + \cdots + \frac{1}{N} = \ln N + \gamma + (\text{something tending to } 0),$$

where $\gamma = 0.5772\dots$ is the Euler–Mascheroni constant. Nobody knows whether γ is rational. You will prove the formula with integrals in first-year analysis, and the harmonic numbers turn up again in Section 9.11 and in Section 11.12.

Most numbers contain a 2

Why does Kempner’s series converge? Because “numbers without a 2” are very rare among large numbers: of the numbers below 10^d , the proportion with no 2 is at most $(0.9)^d$, which tends to 0. Among 100-digit numbers, fewer than 0.003% avoid the digit 2. So we are not removing a tenth of the terms, but almost all of them. Our bound of 80 is generous: in 1979 Robert Baillie used an accelerated method to compute the sum to many decimal places, and it is about 19.257.

Sums of reciprocal powers

The sum in part (ii) is $\zeta(3)$, a value of the Riemann zeta function we met in Section 1.2. In Section 2.4 we will find $\zeta(2) = \pi^2/6$ exactly, and Euler found a formula for $\zeta(4), \zeta(6), \dots$. But nobody has found a formula for $\zeta(3)$. In 1978 Roger Apéry stunned the mathematical world by proving that $\zeta(3)$ is irrational, using integrals not unlike the ones in Section 2.9. Whether $\zeta(5)$ is irrational is still unknown.

The thinking skill

Comparison by blocks

To decide whether a series of positive terms converges, group the terms into blocks on which they are roughly constant, and compare each block with a simple series.

Comparison tests are the bread and butter of first-year analysis. The skill is choosing the comparison: here, blocks of size 2^n or 10^d in which the terms change by at most a fixed factor.

Try this next

- 1 Feel the slowness.** Using $\sum_{r \leq N} 1/r \approx \ln N + 0.5772$, estimate how many terms of the harmonic series you need to pass 10, and to pass 100.

- 2 The test in action.** Use the method of part (i) with $f(r) = 1/r^2$ to show that $\sum 1/r^2 \leq 2$. Then improve the bound by treating the first term separately.
- 3 Between convergence and divergence.** Use the block inequality with $k = 2$ to show that $\sum_{r \geq 2} \frac{1}{r \ln r}$ diverges and $\sum_{r \geq 2} \frac{1}{r(\ln r)^2}$ converges.
- 4 Excluding more.** **HARDER** Show that the sum of the reciprocals of the positive integers with no 2 and no 7 in their decimal representation is less than 35. (Can you see where 35 comes from?)

Further reading

- **Julian Havil, *Gamma: Exploring Euler's Constant*** (Princeton University Press). The harmonic series, γ and much more, written for a strong school student.
- **Mary Hart, *Guide to Analysis*** (Palgrave). A concise first-year text with a clear chapter on series and comparison tests.
- **Robert Baillie, "Sums of reciprocals of integers missing a given digit", *American Mathematical Monthly* 86 (1979).** A short paper you can read with this section behind you.

2.4 $\pi^2/6$

The question

(i) Use De Moivre's theorem to show that, if $\sin \theta \neq 0$, then

$$\frac{(\cot \theta + i)^{2n+1} - (\cot \theta - i)^{2n+1}}{2i} = \frac{\sin(2n+1)\theta}{\sin^{2n+1} \theta},$$

for any positive integer n .

Deduce that the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \cdots + (-1)^n = 0$$

are

$$x = \cot^2 \left(\frac{m\pi}{2n+1} \right)$$

where $m = 1, 2, \dots, n$.

(ii) Hence show that

$$\sum_{m=1}^n \cot^2 \left(\frac{m\pi}{2n+1} \right) = \frac{n(2n-1)}{3}.$$

(iii) Given that $0 < \sin \theta < \theta < \tan \theta$ for $0 < \theta < \frac{1}{2}\pi$, show that

$$\cot^2 \theta < \frac{1}{\theta^2} < 1 + \cot^2 \theta.$$

Hence show that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

The hook

In 1650 Pietro Mengoli asked a simple question: what is

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots?$$

The terms shrink fast enough for the sum to converge (Section 2.3 showed it is at most 2), but nobody could find its value. The Bernoulli brothers tried and failed; Jacob Bernoulli publicly begged for help. The problem became known as the *Basel problem*,

after the Bernoullis' home city.

In 1734 the young Leonhard Euler solved it, and the answer made him famous: the sum is $\pi^2/6$. Why on earth should π , a number about circles, appear in a sum of reciprocal squares? This question gives a complete, rigorous proof using nothing beyond A-level: De Moivre's theorem, the sum of the roots of a polynomial, and a squeeze.

Before you start

The overall plan is worth seeing before the details.

- ▶ Part (i) builds a polynomial whose roots are the numbers $\cot^2\left(\frac{m\pi}{2n+1}\right)$.
- ▶ Part (ii) adds up those roots without knowing them individually, using the coefficients.
- ▶ Part (iii) uses the fact that $\cot \theta \approx 1/\theta$ for small θ to turn a sum of $\cot^2$ values into a sum of $1/m^2$ values, and squeezes.



Ask yourself If a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ has roots $r_1, \dots, r_n$, what is $r_1 + \cdots + r_n$ in terms of the coefficients?

It is $-a_{n-1}/a_n$. That fact is the engine of part (ii).

Working it through

STAGE 1 De Moivre



Ask yourself Write $\cot \theta + i$ as a single fraction. What do you notice about the numerator?

$\cot \theta + i = \frac{\cos \theta + i \sin \theta}{\sin \theta}$, and similarly $\cot \theta - i = \frac{\cos \theta - i \sin \theta}{\sin \theta}$. By De Moivre's theorem,

$$(\cot \theta \pm i)^{2n+1} = \frac{\cos(2n+1)\theta \pm i \sin(2n+1)\theta}{\sin^{2n+1} \theta}.$$

Subtracting, the cosines cancel and the sines double, and dividing by $2i$ gives

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1} \theta}. \quad \blacksquare$$

STAGE 2 The polynomial

Now expand the left-hand side with the binomial theorem instead. Write $c = \cot \theta$ and $N = 2n + 1$.

Ask yourself In $(c+i)^N - (c-i)^N$, which terms of the two binomial expansions cancel, and which double?

The term containing i^j appears with a + sign in the first expansion and with $(-1)^j$ in the second. So terms with even j cancel and terms with odd j double:

$$(c+i)^N - (c-i)^N = 2 \sum_{j \text{ odd}} \binom{N}{j} c^{N-j} i^j.$$

With $j = 2k+1$, $i^j = i(-1)^k$ and $c^{N-j} = c^{2n-2k} = (c^2)^{n-k}$. Dividing by $2i$,

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = \binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \cdots + (-1)^n,$$

where $x = \cot^2 \theta$.

Pause and try For which θ is the left-hand side zero? Use this to find n different roots of the polynomial.

The left-hand side is zero when $\sin(2n+1)\theta = 0$ but $\sin \theta \neq 0$, for instance when $\theta = \frac{m\pi}{2n+1}$ for $m = 1, 2, \dots, n$. These angles all lie strictly between 0 and $\frac{\pi}{2}$, where $\cot$ is positive and strictly decreasing, so the n values $\cot^2(\frac{m\pi}{2n+1})$ are all different. A polynomial of degree n has at most n roots, so these are *all* of them. ■

STAGE 3 Adding the roots

The sum of the roots is $-(\text{coefficient of } x^{n-1})/(\text{coefficient of } x^n)$:

$$\sum_{m=1}^n \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)(2n)(2n-1)}{6(2n+1)} = \frac{n(2n-1)}{3}. \quad \blacksquare$$

Check with $n = 1$: $\cot^2(\pi/3) = \frac{1}{3}$, and $\frac{1 \times 1}{3} = \frac{1}{3}$.

STAGE 4 The squeeze

From $0 < \sin \theta < \theta < \tan \theta$, take reciprocals (which reverses the inequalities) and square:

$$\cot^2 \theta < \frac{1}{\theta^2} < \frac{1}{\sin^2 \theta} = 1 + \cot^2 \theta.$$

Ask yourself Put $\theta = \frac{m\pi}{2n+1}$ and add up over $m = 1, \dots, n$. What do the two outer sums become? What does the middle sum have to do with $\sum 1/m^2$?

The middle sum is $\sum_{m=1}^n \frac{(2n+1)^2}{m^2 \pi^2}$. By part (ii) the left sum is $\frac{n(2n-1)}{3}$, and the right sum

is $n + \frac{n(2n-1)}{3} = \frac{2n(n+1)}{3}$. Multiplying through by $\frac{\pi^2}{(2n+1)^2}$:

$$\frac{\pi^2}{3} \cdot \frac{n(2n-1)}{(2n+1)^2} < \sum_{m=1}^n \frac{1}{m^2} < \frac{\pi^2}{3} \cdot \frac{2n(n+1)}{(2n+1)^2}.$$

As $n \rightarrow \infty$, both $\frac{n(2n-1)}{(2n+1)^2}$ and $\frac{2n(n+1)}{(2n+1)^2}$ tend to $\frac{2}{4} = \frac{1}{2}$ (divide top and bottom by n^2). So both bounds tend to $\frac{\pi^2}{6}$, and the partial sums, trapped between them, must tend to $\frac{\pi^2}{6}$ too (Figure 2.4.1). ■

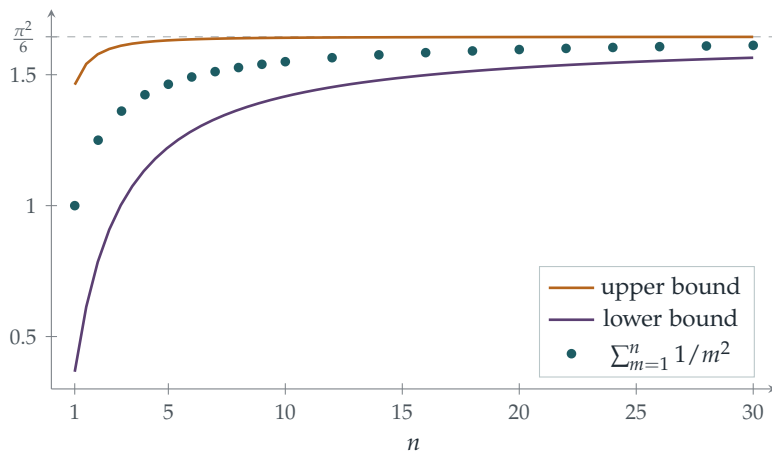


Figure 2.4.1 The squeeze in part (iii). The partial sums of $\sum 1/m^2$ (teal points) are trapped between the lower bound (plum) and the upper bound (amber), and both bounds tend to $\pi^2/6 \approx 1.645$.

Looking back

- **Information from coefficients.** We never found the roots $\cot^2(m\pi/(2n+1))$ in any simpler form, but we added them up exactly.
- **Approximate, then squeeze.** $\cot\theta$ is only approximately $1/\theta$, but the error was controlled by exact inequalities in both directions.
- **Finite to infinite.** Every step until the last concerned finite sums. The infinite sum appeared only when we let $n \rightarrow \infty$ in an inequality, which is the safe way to handle infinity.

What's really going on

Euler's bold argument

Euler's own 1734 argument was not rigorous by modern standards, but it was dazzling. A polynomial with roots $\pm a, \pm b, \pm c, \dots$ and value 1 at 0 factorises as $(1 - x^2/a^2)(1 - x^2/b^2) \dots$. The function $\frac{\sin x}{x}$ has value 1 at 0 and roots $\pm\pi, \pm 2\pi, \pm 3\pi, \dots$, so, treating it as an "infinite polynomial", Euler wrote

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots$$

The coefficient of x^2 on the right is $-\frac{1}{\pi^2} \left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right)$, while from the Maclaurin series $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \dots$ it is $-\frac{1}{6}$. Comparing gives $\pi^2/6$. Our STEP proof is the finite, rigorous version of the same idea: a polynomial whose roots are tied to multiples of π , and its second coefficient.

Beyond reciprocal squares

Euler went on to find $\sum 1/m^4 = \pi^4/90$, $\sum 1/m^6 = \pi^6/945$, and in general $\sum 1/m^{2k}$ is a rational multiple of π^{2k} , involving the *Bernoulli numbers*. The odd powers are a different story: as we saw in Section 2.3, nobody knows a formula for $\sum 1/m^3$. These sums are the values $\zeta(2), \zeta(4), \dots$ of the Riemann zeta function, whose behaviour for complex arguments controls the distribution of the primes (Section 1.2).

Where you will see it again

At university you will meet at least three more proofs of $\zeta(2) = \pi^2/6$: one from *Fourier series* (writing a function as a sum of sines and cosines), one from a double integral, and one from complex analysis. And the number turns up in probability: the chance that two randomly chosen whole numbers have no common factor is $6/\pi^2 \approx 0.61$. (A prime p divides both with probability $1/p^2$, so the chance that no prime divides both is $\prod_p (1 - 1/p^2)$, and Euler showed this product equals $1/\zeta(2)$.) A different appearance of π in a probability limit is the subject of Section 12.23.

The thinking skill

The squeeze, and symmetric functions of roots

To find a limit, trap the quantity between two simpler expressions with the same limit.

And when you need the sum (or product) of some awkward numbers, try to find a polynomial they are the roots of: the coefficients will give you symmetric

combinations of them for free.

Try this next

- 1 The 1992 version.** (From STEP III, 1992.) Using the polynomial from part (i), show that

$$\sum_{k=1}^n \tan^2 \left(\frac{k\pi}{2n+1} \right) = n(2n+1) \quad \text{and} \quad \sum_{k=1}^n \operatorname{cosec}^2 \left(\frac{k\pi}{2n+1} \right) = \frac{2n(n+1)}{3}.$$

(Hint for the first: the numbers $\tan^2$ are the reciprocals of the roots. What polynomial has roots $1/r_i$?)

- 2 Odd squares.** Deduce from $\sum 1/m^2 = \pi^2/6$ that $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}$.
- 3 Fourth powers.** **HARDER** Use $\sum r_i^2 = (\sum r_i)^2 - 2 \sum_{i < j} r_i r_j$ and the coefficients of the polynomial to find $\sum_{m=1}^n \cot^4 \left(\frac{m\pi}{2n+1} \right)$, and adapt the squeeze to prove $\sum 1/m^4 = \pi^4/90$.
- 4 How fast?** Using the lower bound in part (iii), show that $\frac{\pi^2}{6} - \sum_{m=1}^n \frac{1}{m^2} < \frac{\pi^2}{4n}$. How large an n does this guarantee is enough for the partial sum to be within 0.001 of $\pi^2/6$? Compare with the true error, which is close to $1/n$.

Further reading

- **William Dunham, *Euler: The Master of Us All*** (Mathematical Association of America). Chapter 3 tells the story of the Basel problem with the mathematics in full.
- **Aigner & Ziegler, *Proofs from THE BOOK***, Chapter 9, “Three times $\pi^2/6$ ”. Includes the proof in this section.
- **Paul Nahin, *Dr. Euler’s Fabulous Formula*** (Princeton University Press). Complex numbers, Fourier series and sums like this one.

2.5 Viète's Product

The question

You may assume that all infinite sums and products in this question converge.

(i) Prove by induction that for all positive integers n ,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} \cdot \frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\frac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \cdots$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\frac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1 + 2^{\frac{1}{2}}}{2 \times 2^{\frac{1}{4}}}.$$

Use part (ii) to show that

$$\frac{1}{\ln 2} = \frac{1 + 2^{\frac{1}{2}}}{2} \times \frac{1 + 2^{\frac{1}{4}}}{2} \times \frac{1 + 2^{\frac{1}{8}}}{2} \cdots$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2 + \sqrt{2}}}{2} \times \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$

The hook

The last line of this question is one of the most beautiful formulas in mathematics, and one of the oldest of its kind. It was published in 1593 by the French lawyer and mathematician François Viète, and it was the first time anyone had written π *exactly* as the result of an infinite process. Before Viète, π could only be approximated; after him, it could be defined by a formula that, carried on for ever, gives every digit.

The formula looks mysterious: where could nested square roots of 2 come from? The question reveals that it comes from repeatedly using the double-angle formula, and it gets a second, less famous formula, for $\ln 2$, from the same machinery along the way.

Before you start



Ask yourself What is the double-angle formula for $\sinh$? (If you are not sure, write $\sinh 2y$ and $\sinh y \cosh y$ in terms of exponentials and compare.)

$\sinh 2y = 2 \sinh y \cosh y$, exactly like $\sin 2y = 2 \sin y \cos y$. That one identity, used over and over, drives the whole question. The question works with hyperbolic functions until the last part, where we will switch to ordinary trigonometry.

Working it through

STAGE 1 Halving repeatedly

The case $n = 1$ is just the double-angle formula with $y = x/2$:

$$\sinh x = 2 \cosh\left(\frac{x}{2}\right) \sinh\left(\frac{x}{2}\right).$$



Ask yourself Suppose the formula holds for n . The last factor is $\sinh(x/2^n)$. How can you rewrite it to get the formula for $n + 1$?

Apply the double-angle formula to it: $\sinh(x/2^n) = 2 \cosh(x/2^{n+1}) \sinh(x/2^{n+1})$. Substituting,

$$\sinh x = 2^{n+1} \cosh\left(\frac{x}{2}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \cosh\left(\frac{x}{2^{n+1}}\right) \sinh\left(\frac{x}{2^{n+1}}\right),$$

which is the formula for $n + 1$. By induction it holds for all n . ■

For the deduction, divide both sides by x , and then multiply both sides by $\frac{x/2^n}{\sinh(x/2^n)}$.

On the right, $2^n \sinh(x/2^n) \cdot \frac{x/2^n}{x \sinh(x/2^n)} = 1$, leaving just the product of $\cosh$ terms. ■

STAGE 2 The limit

From the series, for $y \neq 0$,

$$\frac{\sinh y}{y} = 1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \dots,$$

which tends to 1 as $y \rightarrow 0$, so its reciprocal $y / \sinh y$ also tends to 1. (If you want more care: every factorial in the tail $\frac{y^2}{3!} + \frac{y^4}{5!} + \dots$ is at least 6, so for $0 < |y| < 1$ the tail is at most $\frac{1}{6}(y^2 + y^4 + \dots) = \frac{y^2}{6(1-y^2)}$, which tends to 0.)

Now let $n \rightarrow \infty$ in part (i), with x fixed. Then $y = x/2^n \rightarrow 0$, so the left-hand side tends to $\frac{\sinh x}{x} \times 1$. The right-hand side is the partial product, which therefore tends to $\frac{\sinh x}{x}$. That is what the infinite product means. ■

STAGE 3 A formula for $\ln 2$

With $x = \ln 2$, $e^{x/2} = \sqrt{2}$, so

$$\cosh\left(\frac{x}{2}\right) = \frac{2^{1/2} + 2^{-1/2}}{2} = \frac{2+1}{2\sqrt{2}} = \frac{3}{2\sqrt{2}}.$$

Similarly $e^{x/4} = 2^{1/4}$, so $\cosh(x/4) = \frac{2^{1/4} + 2^{-1/4}}{2}$, and multiplying top and bottom by $2^{1/4}$ gives $\frac{2^{1/2} + 1}{2 \times 2^{1/4}}$. ■



Ask yourself What is the general pattern for $\cosh(x/2^k)$? What happens when you multiply all the denominators $2 \times 2^{1/2^k}$ together?

The same calculation gives, for every $k \geq 1$,

$$\cosh\left(\frac{x}{2^k}\right) = \frac{1 + 2^{1/2^{k-1}}}{2} \times 2^{-1/2^k}.$$

(For $k = 1$ this is $\frac{1+2}{2} \times 2^{-1/2} = \frac{3}{2\sqrt{2}}$, as before.) Multiplying over $k = 1, 2, 3, \dots$, the powers of 2 combine to $2^{-(1/2+1/4+1/8+\dots)} = 2^{-1}$, and the first fraction is $\frac{1+2}{2} = \frac{3}{2}$. So the infinite product is

$$\frac{3}{2} \times \frac{1}{2} \times \frac{1+2^{1/2}}{2} \times \frac{1+2^{1/4}}{2} \times \frac{1+2^{1/8}}{2} \dots$$

On the other side, $\sinh(\ln 2) = \frac{2-1}{2} = \frac{3}{4}$, so $\frac{\sinh x}{x} = \frac{3}{4 \ln 2}$. The factor $\frac{3}{4}$ cancels from both sides, leaving

$$\frac{1}{\ln 2} = \frac{1+2^{1/2}}{2} \times \frac{1+2^{1/4}}{2} \times \frac{1+2^{1/8}}{2} \dots \quad \blacksquare$$

STAGE 4 Viète's formula

Ask yourself Part (iv) has π in it, so it must come from ordinary trigonometry, not hyperbolic functions. Does everything in parts (i) and (ii) still work with $\sin$ and $\cos$ in place of $\sinh$ and $\cosh$?

It does. The induction used only the double-angle formula, and $\sin 2y = 2 \sin y \cos y$. The limit used only $\frac{\sin y}{y} \rightarrow 1$ as $y \rightarrow 0$, which follows from $\sin y = y - \frac{y^3}{3!} + \dots$ in the same way. So for $x \neq 0$ (and x not a multiple of π , so that no factor is zero along the way),

$$\frac{\sin x}{x} = \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \dots$$

Pause and try Choose x so that the left-hand side is $2/\pi$, and work out the first few factors. The half-angle formula $\cos \frac{\theta}{2} = \sqrt{\frac{1+\cos \theta}{2}}$ (for $0 \leq \theta \leq \pi$) will be useful.

Take $x = \frac{\pi}{2}$. Then $\frac{\sin x}{x} = \frac{1}{\pi/2} = \frac{2}{\pi}$, and the factors are $\cos \frac{\pi}{4}, \cos \frac{\pi}{8}, \cos \frac{\pi}{16}, \dots$. The first is $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. If a factor is $\cos \theta = \frac{c}{2}$, the half-angle formula makes the next one

$$\cos \frac{\theta}{2} = \sqrt{\frac{1+c/2}{2}} = \frac{\sqrt{2+c}}{2}.$$

So the numerators go $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$, each obtained by adding 2 to the previous one and taking the square root. That is Viète's formula. ■

Looking back

- **Telescoping a product.** The identity in part (i) is a product that collapses, just as some sums telescope.
- **Finite identity first, limit second.** As in Section 2.4, every step was exact for finite n , and the only limit needed was the very simple $\frac{\sin y}{y} \rightarrow 1$.
- **Hyperbolic and circular functions are twins.** The same proof worked for both, because both satisfy the same double-angle formula and the same small-angle limit. This is no coincidence: $\cosh(ix) = \cos x$, as you will see in Section 5.7.

What's really going on

Archimedes' polygons

Viète's formula has a geometric meaning that goes back to Archimedes. A regular polygon with 2^n sides inscribed in a circle of radius 1 has area $A_n = 2^{n-1} \sin \frac{2\pi}{2^n}$.

Doubling the number of sides multiplies the area by

$$\frac{A_{n+1}}{A_n} = \frac{2^n \sin(\pi/2^n)}{2^{n-1} \cdot 2 \sin(\pi/2^n) \cos(\pi/2^n)} = \frac{1}{\cos(\pi/2^n)}.$$

Starting from the square ($A_2 = 2$) and doubling for ever, the area tends to the area of the circle, π . So $\pi = 2 \div (\cos \frac{\pi}{4} \cos \frac{\pi}{8} \cos \frac{\pi}{16} \cdots)$, which is Viète's formula. Each factor in the product is the ratio of the areas of two successive polygons in Figure 2.5.1.

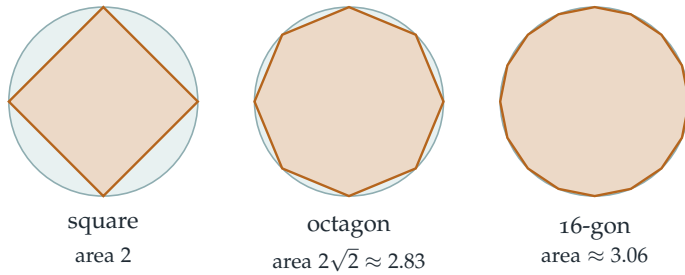


Figure 2.5.1 Regular polygons with 4, 8 and 16 sides inscribed in a circle of radius 1. Each doubling of the number of sides divides the area by one more factor of Viète's product, and the areas increase towards π .

Infinite products

Viète's product was the first infinite product in mathematics. Others followed: John Wallis found $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$ in 1655, and Euler's product for $\frac{\sin x}{x}$ appeared in Section 2.4. At university you will learn when an infinite product converges. Roughly, $\prod(1 + a_n)$ with small positive a_n behaves like $\sum a_n$, because $\ln(1 + a_n) \approx a_n$. Here $1 - \cos(\pi/2^n) \approx \frac{1}{2}(\pi/2^n)^2$, which shrinks by a factor of 4 each time, so the product converges quickly.

How quickly?

After n factors the partial product equals $\frac{1}{2^n \sin(\pi/2^{n+1})}$ exactly, and its error is about $\frac{\pi^2}{6 \cdot 4^{n+1}}$ times $\frac{2}{\pi}$. Each new factor makes the answer about 4 times more accurate, gaining about 0.6 of a decimal digit. Ten factors give $2/\pi$ correct to six decimal places. That is good, but nothing like as fast as the methods of Section 2.6, which double the number of correct digits at each step.

The thinking skill

Iterating an identity

An identity that relates $f(x)$ to $f(x/2)$ can be applied again and again; the result, after a limit, is often an infinite sum or product.

Look for the “remainder” term after n steps (here $2^n \sinh(x/2^n)$) and find its limit separately. This pattern appears all over analysis, from infinite products to the construction of fractals.

Try this next

- 1 The 1990 version.** (From STEP I, 1990.) Show directly, using only the double-angle formula for $\sin$, that

$$\cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{4}\right) \cdots \cos\left(\frac{\alpha}{2^n}\right) = \frac{\sin \alpha}{2^n \sin(\alpha/2^n)},$$

and deduce Viète’s formula in the form $\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2}} \cdots}$.

- 2 Another angle.** Use $x = \frac{\pi}{3}$ in the product for $\frac{\sin x}{x}$ to write $\frac{3\sqrt{3}}{2\pi}$ as an infinite product, and simplify the first two factors.
- 3 How many factors?** Use a calculator to find how many factors of Viète’s product are needed to give π correct to 4 decimal places. Explain why each extra factor gains about 0.6 of a digit.
- 4 A formula for $\ln 3$.** Repeat part (iii) with $x = \ln 3$ to write $\frac{1}{\ln 3}$ as an infinite product. Check numerically that the first five factors give $\frac{1}{\ln 3}$ to two decimal places.

Further reading

- **Petr Beckmann, *A History of π*** (St Martin’s Press). Archimedes, Viète, Wallis and the rest, told with gusto.
- **Jonathan Borwein & Peter Borwein, *Pi and the AGM*** (Wiley). The serious mathematics of computing π , including the method of Section 2.6. Chapter 1 is accessible with this chapter behind you.
- **Eli Maor, *Trigonometric Delights*** (Princeton University Press). Contains a chapter on Viète and his product.

2.6 The Arithmetic–Geometric Mean

The question

In this question, the following theorem may be used without proof.

Let $u_1, u_2, \dots$ be a sequence of real numbers. If the sequence is

- ▶ bounded above, so $u_n \leq b$ for all n , where b is some fixed number
- ▶ and increasing, so $u_n \leq u_{n+1}$ for all n

then there is a number $L \leq b$ such that $u_n \rightarrow L$ as $n \rightarrow \infty$.

For positive real numbers x and y , define $a(x, y) = \frac{1}{2}(x + y)$ and $g(x, y) = \sqrt{xy}$. Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geq 0$,

$$x_{n+1} = a(x_n, y_n), \quad y_{n+1} = g(x_n, y_n).$$

- (i) By considering $(\sqrt{x_n} - \sqrt{y_n})^2$, show that $y_{n+1} < x_{n+1}$, for $n \geq 0$. Show further that, for $n \geq 0$,

- ▶ $x_{n+1} < x_n$
- ▶ $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \rightarrow M$ as $n \rightarrow \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and hence that $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$.

Explain why x_n also tends to M as $n \rightarrow \infty$.

- (ii) Let

$$I(p, q) = \int_0^\infty \frac{1}{\sqrt{(p^2 + x^2)(q^2 + x^2)}} dx,$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2} \left(x - \frac{pq}{x} \right)$ in the integral

$$\int_{-\infty}^\infty \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} dt,$$

that

$$I(p, q) = I(a(p, q), g(p, q)).$$

Hence evaluate $I(x_0, y_0)$ in terms of M .

The hook

Take two positive numbers, say $\sqrt{2}$ and 1. Replace them by their arithmetic mean and their geometric mean. Do it again, and again. Here is what happens:

n	x_n (arithmetic)	y_n (geometric)
0	1.4142135623730950488	1
1	1.2071067811865475244	1.1892071150027210667
2	1.1981569480946342956	1.1981235214931201226
3	1.1981402347938772091	1.1981402346773072058
4	1.1981402347355922074	1.1981402347355922074

After four steps the two numbers agree to twenty decimal places. The common limit is the *arithmetic–geometric mean* $M(\sqrt{2}, 1) = 1.19814\dots$

On 30 May 1799 the 21-year-old Carl Friedrich Gauss wrote in his diary that he had noticed $1/M(\sqrt{2}, 1)$ agreed, to eleven decimal places, with a constant that had appeared in a completely different problem: the arc length of a curve called the lemniscate. “This will surely open up an entirely new field of analysis,” he wrote. It did. This question proves the key fact behind Gauss’s discovery: the AGM evaluates an integral that cannot be done by any of the usual methods.

Before you start



Ask yourself Why is the arithmetic mean of two different positive numbers always bigger than their geometric mean? Can you prove it in one line?

That is the AM–GM inequality, and the hint in part (i) gives the one-line proof: $(\sqrt{x} - \sqrt{y})^2 > 0$ when $x \neq y$, which expands to $x + y > 2\sqrt{xy}$. Everything in part (i) follows from this and from the fact that each mean lies between the two numbers it comes from.

Working it through

STAGE 1 The two sequences squeeze together

If $y_n < x_n$ then $\sqrt{x_n} \neq \sqrt{y_n}$, so $x_n - 2\sqrt{x_n y_n} + y_n > 0$, which says $x_{n+1} > y_{n+1}$. Since $y_0 < x_0$, induction gives $y_n < x_n$ for all n .



Ask yourself Why is the arithmetic mean of x_n and y_n less than x_n , and the geometric mean bigger than y_n ?

Because $y_n < x_n$: $x_{n+1} = \frac{x_n + y_n}{2} < \frac{x_n + x_n}{2} = x_n$, and $y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n y_n} = y_n$. So

$$y_0 < y_1 < y_2 < \dots < x_2 < x_1 < x_0.$$

The y_n are increasing and bounded above by x_0 , so by the theorem they tend to a limit M .

For the gap,

$$x_{n+1} - y_{n+1} < x_{n+1} - y_n = \frac{x_n + y_n}{2} - y_n = \frac{1}{2}(x_n - y_n),$$

and it is positive by the first part. So the gap at least halves each time: $0 < x_n - y_n < (x_0 - y_0)/2^n$, which tends to 0.

Finally $x_n = y_n + (x_n - y_n)$ is the sum of a sequence tending to M and a sequence tending to 0, so $x_n \rightarrow M$ too. ■

STAGE 2 The substitution

The substitution looks unmotivated, but it is designed so that both brackets under the square root become perfect squares (or nearly).

Ask yourself With $t = \frac{1}{2} \left(x - \frac{pq}{x} \right)$, compute $t^2 + pq$. Then compute $t^2 + \frac{1}{4}(p + q)^2$. Do either of them factorise nicely?

First,

$$t^2 + pq = \frac{1}{4} \left(x^2 - 2pq + \frac{p^2 q^2}{x^2} \right) + pq = \frac{1}{4} \left(x + \frac{pq}{x} \right)^2.$$

Second,

$$\begin{aligned} t^2 + \frac{1}{4}(p + q)^2 &= \frac{1}{4} \left(x^2 - 2pq + \frac{p^2 q^2}{x^2} + p^2 + 2pq + q^2 \right) \\ &= \frac{x^4 + (p^2 + q^2)x^2 + p^2 q^2}{4x^2} = \frac{(x^2 + p^2)(x^2 + q^2)}{4x^2}. \end{aligned}$$

And $\frac{dt}{dx} = \frac{1}{2} \left(1 + \frac{pq}{x^2} \right) = \frac{x^2 + pq}{2x^2}.$

Pause and try Put these together. As x runs from 0 to ∞ , what does t do? What does the t -integral become?

As x increases from 0 to ∞ , t increases (its derivative is positive) from $-\infty$ to ∞ . For $x > 0$,

$$\sqrt{\left(t^2 + \frac{1}{4}(p + q)^2\right)(t^2 + pq)} = \frac{\sqrt{(x^2 + p^2)(x^2 + q^2)}}{2x} \cdot \frac{x^2 + pq}{2x},$$

so

$$\frac{dt}{\sqrt{\dots}} = \frac{x^2 + pq}{2x^2} \cdot \frac{4x^2}{(x^2 + pq)\sqrt{(x^2 + p^2)(x^2 + q^2)}} dx = \frac{2 dx}{\sqrt{(x^2 + p^2)(x^2 + q^2)}}.$$

So the t -integral equals $2I(p, q)$.

On the other hand, the t -integral has exactly the form of I with p and q replaced by $\frac{1}{2}(p+q) = a(p,q)$ and $\sqrt{pq} = g(p,q)$, except that it runs over the whole real line. The integrand is even in t , so the t -integral is $2I(a(p,q), g(p,q))$. Comparing,

$$I(p,q) = I(a(p,q), g(p,q)). \quad \blacksquare$$

STAGE 3 Evaluating the integral

? **Ask yourself** Apply this identity repeatedly, starting from (x_0, y_0) . What is $I(x_0, y_0)$ equal to after n steps? What happens as $n \rightarrow \infty$?

$I(x_0, y_0) = I(x_1, y_1) = I(x_2, y_2) = \cdots = I(x_n, y_n)$ for every n . As $n \rightarrow \infty$, both arguments tend to M , and

$$I(M, M) = \int_0^\infty \frac{dx}{M^2 + x^2} = \left[\frac{1}{M} \arctan \frac{x}{M} \right]_0^\infty = \frac{\pi}{2M}.$$

So

$$I(x_0, y_0) = \frac{\pi}{2M}. \quad \blacksquare$$

The question lets us take the step “ $I(x_n, y_n) \rightarrow I(M, M)$ ” on trust, but it can be justified neatly. Making p or q larger makes the integrand smaller, and $y_n < x_n$, so

$$\frac{\pi}{2x_n} = I(x_n, x_n) < I(x_n, y_n) < I(y_n, y_n) = \frac{\pi}{2y_n}.$$

Both bounds tend to $\frac{\pi}{2M}$, and $I(x_n, y_n)$ is the same number $I(x_0, y_0)$ for every n , so it must equal $\frac{\pi}{2M}$. A squeeze again, as in Section 2.4.

Looking back

- **Two monotone sequences squeezing together.** One goes up, one goes down, and the gap shrinks to zero. This is the cleanest way to prove two sequences have a common limit.
- **An invariant.** The quantity $I(x_n, y_n)$ never changes from step to step. Finding something that stays fixed while everything else moves is one of the most powerful ideas in mathematics.
- **Clever substitutions are designed, not guessed.** The substitution was chosen so that the expressions under the square root factorised.

What’s really going on

Faster than the question admits

Part (i) shows the gap at least halves at each step, which would give about one extra binary digit per step. The table in the hook shows something far more dramatic: the number of correct digits roughly *doubles* at each step (2, 4, 10, 21, 43, . . .). In fact

$$x_{n+1} - y_{n+1} = \frac{1}{2}(\sqrt{x_n} - \sqrt{y_n})^2 = \frac{(x_n - y_n)^2}{2(\sqrt{x_n} + \sqrt{y_n})^2},$$

so each gap is roughly a constant times the *square* of the previous one. This is called *quadratic convergence*, and we will study it properly in Section 2.7.

Elliptic integrals

Our integral $I(p, q)$ is a disguised *complete elliptic integral*, a type of integral that arises in computing the arc length of an ellipse and that cannot be expressed using the functions you know. Its most famous application is the pendulum. The A-level formula $T = 2\pi\sqrt{L/g}$ for the period is only an approximation for small swings. The exact period of a pendulum released from angle θ_0 is

$$T = \frac{2\pi\sqrt{L/g}}{M(1, \cos \frac{\theta_0}{2})},$$

an elliptic integral evaluated by the AGM. For $\theta_0 = 90^\circ$ the true period is about 18% longer than the simple formula predicts. Gauss’s diary entry concerned the lemniscate, a figure-of-eight curve, whose total arc length involves $\int_0^1 \frac{dt}{\sqrt{1-t^4}}$; that integral equals $I(\sqrt{2}, 1)$ in disguise, so the AGM evaluates it.

Computing π

In 1976 Richard Brent and Eugene Salamin independently turned Gauss’s ideas into an algorithm for computing π that doubles the number of correct digits at every step. Twenty-five steps give tens of millions of digits. Algorithms of this kind were used in record-breaking computations of π in the 1980s and 1990s, and quadratically convergent algorithms like it are now standard wherever high precision is needed. Compare Viète’s product in Section 2.5, which gains about 0.6 of a digit per step.

The thinking skill

Look for an invariant

When a process changes many things at once, look for a quantity that does not change. Evaluate it at the end of the process, where everything is simple.

Here the process was the AGM step and the invariant was the integral I . At the end of the process the two arguments were equal and the integral was elementary. Invariants appear throughout mathematics, from conservation of energy in mechanics (Chapter 10) to the proof that some puzzles are impossible.

Try this next

- 1 **Quadratic convergence.** Prove the identity $x_{n+1} - y_{n+1} = \frac{(x_n - y_n)^2}{2(\sqrt{x_n} + \sqrt{y_n})^2}$, and deduce that $x_{n+1} - y_{n+1} \leq \frac{(x_n - y_n)^2}{8y_0}$.
- 2 **By hand.** Compute $M(2, 1)$ to six decimal places using a calculator. How many steps did you need?
- 3 **Scaling.** Show that $M(\lambda x, \lambda y) = \lambda M(x, y)$ for $\lambda > 0$, and that $I(\lambda p, \lambda q) = \lambda^{-1} I(p, q)$. Check that these two facts are consistent with $I(x_0, y_0) = \pi / (2M)$.
- 4 **The pendulum.** Use the formula above to find, to three significant figures, how much longer the period of a pendulum released from 60° is than the small-angle prediction.

Further reading

- **David Cox**, “The arithmetic–geometric mean of Gauss”, *L’Enseignement Mathématique* 30 (1984). The classic account of the history and mathematics; the first sections need only calculus.
- **Jonathan Borwein & Peter Borwein**, *Pi and the AGM* (Wiley). The definitive book on the AGM and fast computation of π .
- **Jonathan Borwein & Keith Devlin**, *The Computer as Crucible* (A. K. Peters). Experimental mathematics, including discoveries like Gauss’s, made by spotting numerical coincidences.

2.7 Six Digits or Six Hundred

The question

- (i) The sequence x_n for $n = 0, 1, 2, \dots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geq 0$.

- (a) Explain briefly why $x_n \geq 1$ for all n .
 (b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$|x_{n+1}^2 - 2| \leq \frac{1}{4} |x_n^2 - 2|.$$

- (c) Show that

$$2 - 10^{-6} \leq x_{10}^2 \leq 2.$$

- (ii) The sequence y_n for $n = 0, 1, 2, \dots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geq 0$.

- (a) Show that, for $n \geq 0$,

$$y_{n+1} - \sqrt{2} = \frac{(y_n - \sqrt{2})^2}{2y_n}$$

and deduce that $y_n \geq 1$ for $n \geq 0$.

- (b) Show that

$$y_n - \sqrt{2} \leq 2 \left(\frac{\sqrt{2} - 1}{2} \right)^{2^n}$$

for $n \geq 1$.

- (c) Using the fact that $\sqrt{2} - 1 < \frac{1}{2}$, or otherwise, show that

$$\sqrt{2} \leq y_{10} \leq \sqrt{2} + 10^{-600}.$$

The hook

Two simple recipes for approximating $\sqrt{2}$, both starting from 1, both using nothing but addition, multiplication and division. After ten steps, the first is correct to about 8 decimal places. The second is correct to more than 700.

The difference is not that the second recipe is cleverer about $\sqrt{2}$. It is a difference in the *shape* of the error: in the first, each step multiplies the error by a fixed fraction; in the second, each step *squares* it. The question makes this precise, and in doing so introduces one of the central ideas of numerical analysis, the subject that underlies every calculator, every weather forecast and every engineering simulation.

Before you start

Compute a few terms of each sequence as fractions.

$$x_n : 1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \dots \quad y_n : 1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \dots$$



Ask yourself Do you notice anything about the two lists? (Look at which fractions appear in both.)

Every y_n appears in the x list, but further and further along: $y_2 = x_3$, $y_3 = x_7$, and in general $y_n = x_{2^n - 1}$. The second recipe is skipping ahead exponentially fast through the same sequence of approximations. Keep this in mind; we will explain it at the end.

Working it through

STAGE 1 The first recipe stays at least 1

We can write $x_{n+1} = 1 + \frac{1}{x_n + 1}$. If $x_n \geq 1$ then $x_n + 1 > 0$, so $x_{n+1} > 1$. Since $x_0 = 1$, induction gives $x_n \geq 1$ for all n . ■

STAGE 2 The error in the first recipe



Ask yourself The question measures the error by $x_n^2 - 2$ rather than $x_n - \sqrt{2}$. Write $x_{n+1}^2 - 2$ as a single fraction in terms of x_n . What do you notice about the numerator?

$$x_{n+1}^2 - 2 = \frac{(x_n + 2)^2 - 2(x_n + 1)^2}{(x_n + 1)^2} = \frac{x_n^2 + 4x_n + 4 - 2x_n^2 - 4x_n - 2}{(x_n + 1)^2} = \frac{2 - x_n^2}{(x_n + 1)^2}.$$

The numerator is exactly $-(x_n^2 - 2)$. So $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite signs (the

approximations alternate either side of $\sqrt{2}$, and since $x_n \geq 1$ makes $(x_n + 1)^2 \geq 4$,

$$|x_{n+1}^2 - 2| = \frac{|x_n^2 - 2|}{(x_n + 1)^2} \leq \frac{1}{4} |x_n^2 - 2|. \quad \blacksquare$$

STAGE 3 Ten steps

Applying this ten times, $|x_{10}^2 - 2| \leq 4^{-10} |x_0^2 - 2| = 4^{-10} = \frac{1}{1048576} < 10^{-6}$. For the sign: $x_0^2 - 2 = -1 < 0$, and the sign alternates, so $x_n^2 - 2 < 0$ for every even n . So $x_{10}^2 < 2$, and together these give $2 - 10^{-6} \leq x_{10}^2 \leq 2$. $\blacksquare$

The error in x_n^2 is multiplied by at most $\frac{1}{4}$ at each step. That is called *linear convergence*: a fixed number of extra correct digits per step (here at least $\log_{10} 4 \approx 0.6$; in fact closer to 0.77).

STAGE 4 The error in the second recipe

? **Ask yourself** Write $y_{n+1} - \sqrt{2}$ as a single fraction. The numerator should remind you of a perfect square.

$$y_{n+1} - \sqrt{2} = \frac{y_n^2 + 2 - 2\sqrt{2}y_n}{2y_n} = \frac{(y_n - \sqrt{2})^2}{2y_n}. \quad \blacksquare$$

If $y_n > 0$, the right-hand side is non-negative, so $y_{n+1} \geq \sqrt{2} > 1$. Since $y_0 = 1$, induction gives $y_n \geq 1$ for all n . $\blacksquare$ (In fact $y_n \geq \sqrt{2}$ for all $n \geq 1$: after the first step, the second recipe always approaches $\sqrt{2}$ from above.)

STAGE 5 Squaring the error

Write $e_n = y_n - \sqrt{2}$, so $e_n \geq 0$ for $n \geq 1$, and $e_{n+1} = \frac{e_n^2}{2y_n} \leq \frac{e_n^2}{2}$ because $y_n \geq 1$.

/ **Pause and try** Prove by induction that $e_n \leq 2r^{2^n}$ for $n \geq 1$, where $r = \frac{\sqrt{2}-1}{2}$. Check the base case carefully.

Base case: $e_1 = \frac{(1 - \sqrt{2})^2}{2 \cdot 1} = \frac{(\sqrt{2} - 1)^2}{2}$, while $2r^2 = 2 \cdot \frac{(\sqrt{2} - 1)^2}{4} = \frac{(\sqrt{2} - 1)^2}{2}$. They are equal, so the base case holds (exactly).

Inductive step: if $e_n \leq 2r^{2^n}$ then

$$e_{n+1} \leq \frac{e_n^2}{2} \leq \frac{4r^{2^{n+1}}}{2} = 2r^{2^{n+1}}. \quad \blacksquare$$

STAGE 6 Six hundred digits

Since $\sqrt{2} - 1 < \frac{1}{2}$, we have $r < \frac{1}{4}$, so

$$0 \leq y_{10} - \sqrt{2} \leq 2 \left(\frac{1}{4} \right)^{2^{10}} = 2 \cdot 2^{-2048} = 2^{-2047}.$$

Now $2^{10} = 1024 > 10^3$, so $2^{-2047} < 2^{-2040} = (2^{10})^{-204} < 10^{-612}$, which is certainly less than 10^{-600} . ■

Looking back

- **Find the error recurrence.** Both parts worked by writing the new error as an exact function of the old error. Everything else followed.
- **Linear versus quadratic.** “New error $\leq$ constant $\times$ old error” gives a fixed number of extra digits per step. “New error $\leq$ constant $\times$ (old error)²” doubles the number of correct digits per step (Figure 2.7.1).
- **Squared quantities control signs.** In part (ii) the error after one step is a square divided by something positive, so it can never be negative: the iteration approaches from above.

What’s really going on

Newton’s method

The second recipe is *Newton’s method* applied to $f(y) = y^2 - 2$. Newton’s method finds a root of f by repeatedly replacing a guess y by the point where the tangent at y crosses the axis:

$$y_{\text{new}} = y - \frac{f(y)}{f'(y)} = y - \frac{y^2 - 2}{2y} = \frac{y^2 + 2}{2y}.$$

Near a simple root α , a Taylor expansion shows that the new error is approximately $\frac{f''(\alpha)}{2f'(\alpha)} \times (\text{old error})^2$, which is why Newton’s method converges quadratically. This particular case, “average y and $2/y$ ”, was known to the Babylonians; a clay tablet from around 1700 BC, now called YBC 7289, gives $\sqrt{2}$ in base 60 with an error of less than one part in a million.

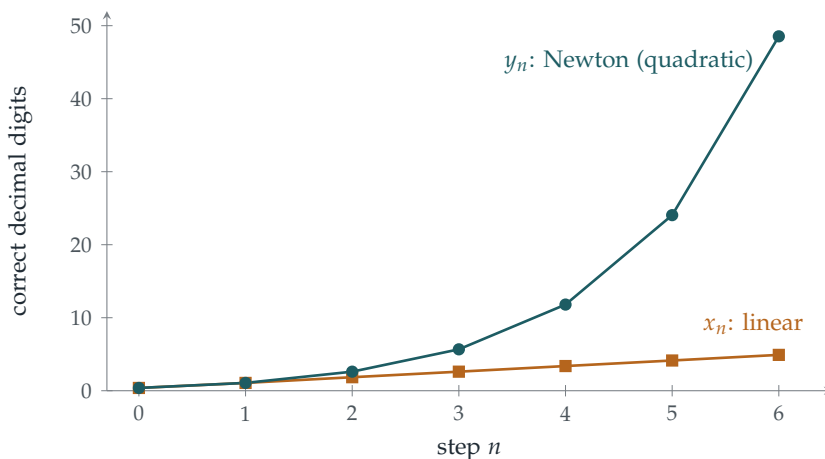


Figure 2.7.1 The number of correct decimal digits of $\sqrt{2}$ after n steps of each recipe. The linear recipe gains about 0.77 digits per step; Newton's method roughly doubles its number of correct digits at every step, reaching about 780 digits by $n = 10$.

Fixed points and their speed

Both recipes have the form $u_{n+1} = g(u_n)$ with $g(\sqrt{2}) = \sqrt{2}$, so $\sqrt{2}$ is a *fixed point*, as in the cobweb diagram of Section 2.2. Near a fixed point α , the error is multiplied by about $g'(\alpha)$ at each step. For the first recipe, $g(x) = \frac{x+2}{x+1}$ and $g'(\sqrt{2}) = -\frac{1}{(\sqrt{2}+1)^2} = -(3 - 2\sqrt{2}) \approx -0.17$: the minus sign explains the alternating signs and 0.17 explains the 0.77 digits per step ($\log_{10}(1/0.17) \approx 0.77$). For Newton's method $g'(\alpha) = 0$, so the linear term vanishes and the error is controlled by the next term, the square. Designing iterations with $g'(\alpha) = 0$ is the art of fast numerical methods.

Why the same fractions appear

The fractions $x_n = p_n/q_n$ satisfy $p_{n+1} = p_n + 2q_n$ and $q_{n+1} = p_n + q_n$, and $p_n^2 - 2q_n^2 = \pm 1$, alternating. These are the solutions of *Pell's equation* $p^2 - 2q^2 = \pm 1$, and they are the convergents of the continued fraction $\sqrt{2} = [1; 2, 2, 2, \dots]$. They come from powers of $1 + \sqrt{2}$: $p_n + q_n\sqrt{2} = (1 + \sqrt{2})^{n+1}$. Newton's step *squares* $p + q\sqrt{2}$ (try it: $(p + q\sqrt{2})^2 = p^2 + 2q^2 + 2pq\sqrt{2}$, and $\frac{p^2+2q^2}{2pq}$ is exactly Newton's formula applied to p/q). Squaring doubles the exponent, which is why $y_n = x_{2^n-1}$. The same pattern of "squaring" drives the iteration in Section 2.8.

The thinking skill

Analyse the error, not the answer

To understand how an approximation behaves, write down how the error at one step depends on the error at the step before.

This is the starting point of numerical analysis. The answer itself ($\sqrt{2}$) is the same for both recipes; what differs, and what matters in practice, is how the error evolves.

Try this next

- 1 Pell's equation.** Show that if $y_n = p/q$ in lowest terms with $p^2 - 2q^2 = 1$, then $y_{n+1} = \frac{p^2 + 2q^2}{2pq}$ also satisfies this equation. Check that $577^2 - 2 \times 408^2 = 1$.
- 2 Cube roots.** Write down Newton's method for $f(y) = y^3 - 2$, and show that the new error is $\frac{(y_n - \alpha)^2(2y_n + \alpha)}{3y_n^2}$, where $\alpha = \sqrt[3]{2}$. Deduce that if $y_n > \alpha$ then the new error is at most $\frac{1}{\alpha}$ times the square of the old error. (Hint: use $\alpha^3 = 2$.)
- 3 Division without dividing.** Show that Newton's method for $f(y) = \frac{1}{y} - a$ gives $y_{n+1} = y_n(2 - ay_n)$, which needs no division. If $e_n = 1 - ay_n$, show that $e_{n+1} = e_n^2$. (Computer processors use iterations like this to divide.)
- 4 When Newton fails.** Apply Newton's method to $f(y) = y^{1/3}$, whose only root is 0. Show that each step doubles the distance from the root, whatever you start with. Which hypothesis of the error formula fails?

Further reading

- **Endre Süli & David Mayers, *An Introduction to Numerical Analysis*** (Cambridge University Press). The standard undergraduate text; Chapter 1 covers fixed-point iteration and Newton's method.
- **Nick Higham (ed.), *The Princeton Companion to Applied Mathematics*** (Princeton University Press). Short articles on Newton's method and numerical analysis, written by experts for a wide audience.
- **David Fowler & Eleanor Robson, "Square root approximations in Old Babylonian mathematics: YBC 7289 in context"**, *Historia Mathematica* 25 (1998).

2.8 Squaring and Subtracting Two

The question

A sequence $x_1, x_2, \dots$ of real numbers is defined by $x_{n+1} = x_n^2 - 2$ for $n \geq 1$ and $x_1 = a$.

- (i) Show that if $a > 2$ then $x_n \geq 2 + 4^{n-1}(a - 2)$.
- (ii) Show also that $x_n \rightarrow \infty$ as $n \rightarrow \infty$ if and only if $|a| > 2$.
- (iii) When $a > 2$, a second sequence $y_1, y_2, \dots$ is defined by

$$y_n = \frac{Ax_1x_2 \cdots x_n}{x_{n+1}},$$

where A is a positive constant and $n \geq 1$.

Prove that, for a certain value of a , with $a > 2$, which you should find in terms of A ,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for all $n \geq 1$.

Determine whether, for this value of a , the second sequence converges.

The hook

“Square it and subtract two” is about the simplest rule you could use to generate a sequence. Yet it divides the real line into two worlds. Start outside $[-2, 2]$ and the numbers explode to infinity, faster than exponentially. Start inside and they stay trapped for ever, wandering in a way that is, in a precise sense, completely unpredictable: one of the simplest examples of *chaos*.

The same rule, carried out with very large whole numbers, is the engine of the Lucas–Lehmer test, the method used to find every one of the largest known prime numbers. This question explores the escaping side of the story and finds a hidden formula that makes the sequence completely solvable.

Before you start

Try some starting values. With $a = 3$: 3, 7, 47, 2207, ... With $a = 1$: 1, -1, -1, -1, ... With $a = 0$: 0, -2, 2, 2, 2, ... With $a = 0.3$: 0.3, -1.91, 1.648, 0.716, -1.487, 0.211, ...

? **Ask yourself** Why can a number in $[-2, 2]$ never escape? What is the largest and smallest that $x^2 - 2$ can be when $-2 \leq x \leq 2$?

If $-2 \leq x \leq 2$ then $0 \leq x^2 \leq 4$, so $-2 \leq x^2 - 2 \leq 2$. The interval $[-2, 2]$ is mapped into itself. That will be half of part (ii).

Working it through

STAGE 1 Escaping

? **Ask yourself** Part (i) has a 4^{n-1} in it, suggesting that the distance above 2 at least quadruples at each step. Write $x_n = 2 + d$. What is x_{n+1} in terms of d ?

$x_{n+1} = (2 + d)^2 - 2 = 2 + 4d + d^2$. So if $d \geq 0$, the new distance above 2 is $4d + d^2 \geq 4d$.

Now induct. When $n = 1$, $x_1 = a = 2 + 4^0(a - 2)$. If $x_n \geq 2 + 4^{n-1}(a - 2)$, then $d = x_n - 2 \geq 4^{n-1}(a - 2) \geq 0$, so

$$x_{n+1} - 2 \geq 4d \geq 4^n(a - 2). \quad \blacksquare$$

STAGE 2 Escape if and only if $|a| > 2$

? **Ask yourself** “If and only if” means two separate proofs. Which direction does part (i) almost give you, and what is missing?

If $|a| > 2$: when $a > 2$, part (i) gives $x_n \geq 2 + 4^{n-1}(a - 2) \rightarrow \infty$. When $a < -2$, the first step fixes the sign: $x_2 = a^2 - 2 > 4 - 2 = 2$, and the sequence from x_2 onwards is the same sequence started at $x_2 > 2$, so it tends to infinity.

Only if: if $|a| \leq 2$, then $a \in [-2, 2]$, and by the observation above every term stays in $[-2, 2]$. A bounded sequence cannot tend to infinity. $\blacksquare$

STAGE 3 A hidden formula

Part (iii) looks forbidding. Before attempting it, it is worth finding a formula for x_n .

? **Ask yourself** If $a > 2$, there is a number $t > 1$ with $a = t + \frac{1}{t}$. (Why?) What are x_2 and x_3 in terms of t ?

The function $t + 1/t$ increases from 2 to ∞ as t goes from 1 to ∞ , so every $a > 2$ can be

written this way. Then

$$x_2 = \left(t + \frac{1}{t}\right)^2 - 2 = t^2 + \frac{1}{t^2}, \quad x_3 = \left(t^2 + \frac{1}{t^2}\right)^2 - 2 = t^4 + \frac{1}{t^4},$$

and by induction

$$x_n = t^{2^{n-1}} + t^{-2^{n-1}}.$$

Squaring and subtracting 2 simply doubles the exponent. Now the question becomes manageable.

STAGE 4 The product telescopes

Ask yourself What is $\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right)$? What happens if you multiply the result by $t^2 + \frac{1}{t^2}$?

$\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right) = t^2 - \frac{1}{t^2}$, and then $\left(t^2 - \frac{1}{t^2}\right)\left(t^2 + \frac{1}{t^2}\right) = t^4 - \frac{1}{t^4}$, and so on: each factor x_k doubles the exponent in a difference of two powers. So

$$\left(t - \frac{1}{t}\right)x_1x_2\cdots x_n = t^{2^n} - t^{-2^n}.$$

Pause and try Compute $x_{n+1}^2 - 4$ in terms of t , and compare with the product.

$$x_{n+1}^2 - 4 = \left(t^{2^n} + t^{-2^n}\right)^2 - 4 = \left(t^{2^n} - t^{-2^n}\right)^2.$$

Since $t > 1$, $t^{2^n} - t^{-2^n} > 0$, so $\sqrt{x_{n+1}^2 - 4} = t^{2^n} - t^{-2^n} = \left(t - \frac{1}{t}\right)x_1x_2\cdots x_n$. Therefore

$$\frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}} = \frac{\left(t - \frac{1}{t}\right)x_1\cdots x_n}{x_{n+1}},$$

which is y_n for every n exactly when $A = t - \frac{1}{t}$. Finally, $\left(t - \frac{1}{t}\right)^2 = \left(t + \frac{1}{t}\right)^2 - 4 = a^2 - 4$, so the condition is $A = \sqrt{a^2 - 4}$, that is,

$$a = \sqrt{A^2 + 4}.$$

(This is greater than 2, as required, because $A > 0$.) ■

STAGE 5 Does y_n converge?

With this value of a ,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}} = \sqrt{1 - \frac{4}{x_{n+1}^2}}.$$

Since $x_{n+1} \rightarrow \infty$ by part (ii), $\frac{4}{x_{n+1}^2} \rightarrow 0$ and $y_n \rightarrow 1$. The sequence converges, to 1. ■

So the infinite product $x_1 x_2 x_3 \cdots$ grows at exactly the same rate as x_{n+1}/A : we have found the precise growth rate of a product of rapidly growing terms.

Looking back

- **An invariant region.** $[-2, 2]$ is mapped into itself, and that one fact settles half of part (ii).
- **A substitution that linearises.** Writing $a = t + 1/t$ turned “square and subtract 2” into “double the exponent”. Messy nonlinear behaviour became simple.
- **Telescoping again.** As with the products in Section 2.5, a product of many factors collapsed because each factor fitted with the next.

What’s really going on

Order and chaos

What happens inside $[-2, 2]$? There, $a = t + 1/t$ fails (it needs $|a| \geq 2$), but the trigonometric version works: write $a = 2 \cos \theta$. Then $x_2 = 4 \cos^2 \theta - 2 = 2 \cos 2\theta$, and in general

$$x_n = 2 \cos(2^{n-1} \theta).$$

So inside the interval the map simply *doubles an angle*. That sounds harmless, but doubling an angle is chaotic. Write θ/π in binary: doubling shifts the binary point one place to the right, throwing away the leading digit. After 50 steps, x_n depends on the 51st binary digit of θ/π , so an error in the starting value of 10^{-15} (the precision of a typical computer) makes the answer after 50 steps completely unreliable. This sensitive dependence on initial conditions is the hallmark of chaos, illustrated in Figure 2.8.1. The map $x \mapsto x^2 - 2$ is equivalent, by a change of variable, to the famous logistic map $x \mapsto 4x(1 - x)$, which we meet again in Section 8.7.

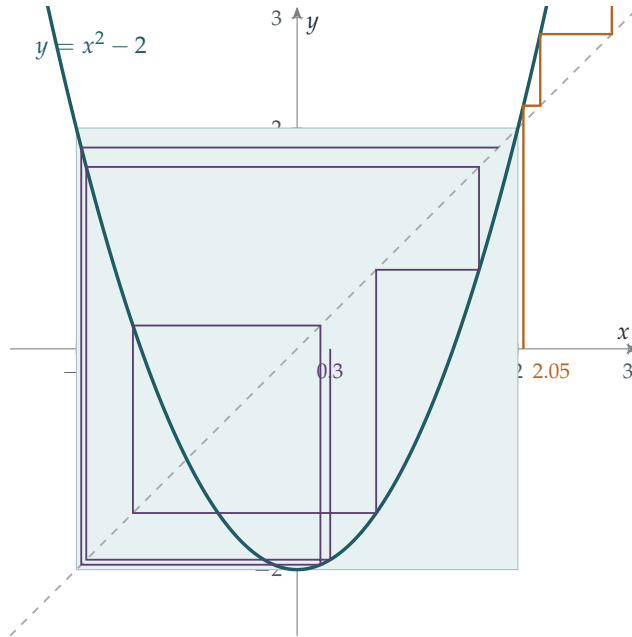


Figure 2.8.1 Cobweb diagrams for $x_{n+1} = x_n^2 - 2$. Starting at 0.3 (plum), the orbit stays in the shaded square $[-2, 2]^2$ for ever, wandering chaotically. Starting just outside, at 2.05 (amber), it escapes: 2.05, 2.20, 2.85, 6.13, ...

Finding the largest primes

A *Mersenne number* is $M_p = 2^p - 1$. In 1876 Édouard Lucas found, and in 1930 Derrick Lehmer completed, a remarkable test for whether M_p is prime (for p an odd prime): start with $s_0 = 4$ and repeatedly square and subtract 2, reducing modulo M_p each time. Then

$$M_p \text{ is prime} \iff s_{p-2} \equiv 0 \pmod{M_p}.$$

For $M_7 = 127$: $s = 4, 14, 194 \equiv 67, 67^2 - 2 \equiv 42, 42^2 - 2 \equiv 111, 111^2 - 2 \equiv 0$, so 127 is prime. The connection with this question is the substitution: $4 = t + 1/t$ with $t = 2 + \sqrt{3}$, so $s_k = (2 + \sqrt{3})^{2^k} + (2 - \sqrt{3})^{2^k}$, and the proof of the test uses arithmetic with numbers of the form $a + b\sqrt{3}$, rather like the Gaussian integers of Section 1.6. Because each step needs only one squaring, the test is extremely fast, and the largest known primes are all Mersenne primes. As of the time of writing the record is $2^{136279841} - 1$, a number with more than 41 million digits, found in 2024.

The Mandelbrot set

Replace -2 by any constant c and allow complex numbers: $z \mapsto z^2 + c$. The set of c for which the orbit of 0 stays bounded is the *Mandelbrot set*. Our map has $c = -2$, the leftmost point of the Mandelbrot set, and the set of starting values whose orbits stay bounded (its *Julia set*) is exactly the interval $[-2, 2]$ that we found in part (ii). For almost every other c , that set is a fractal.

The thinking skill

Change variables to make a map simple

If an iteration looks complicated, look for a substitution that turns it into something simple, like doubling or adding a constant.

Here $x = t + 1/t$ and $x = 2 \cos \theta$ turned $x \mapsto x^2 - 2$ into doubling. In the theory of dynamical systems this is called a *conjugacy*, and finding one is often the key to understanding a system completely.

Try this next

- 1 Inside the interval.** Prove that if $a = 2 \cos \theta$ then $x_n = 2 \cos(2^{n-1}\theta)$. Find all values of a for which $x_3 = a$ but $x_2 \neq a$. (Hint: you want $4\theta \equiv \pm\theta$, modulo 2π .)
- 2 Lucas–Lehmer by hand.** Use the test to show that $M_5 = 31$ is prime and that $M_{11} = 2047$ is not. (For M_{11} you will need nine steps; a calculator helps. Check your conclusion by factorising 2047.)
- 3 An explicit example.** Take $A = \frac{3}{2}$. Find a and t , and check the formula $y_1 = \sqrt{x_2^2 - 4/x_2}$ numerically.
- 4 A sum that telescopes.** **HARDER** With $a = t + 1/t$ and $t > 1$, show that

$$\frac{1}{x_1 x_2 \cdots x_n} = \frac{t - t^{-1}}{t^{2^n} - t^{-2^n}}.$$

Using the identity $\frac{1}{z-1} - \frac{1}{z^2-1} = \frac{z}{z^2-1}$ with $z = t^{2^n}$, deduce that

$$\frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} + \cdots = \frac{1}{t}.$$

Check this numerically when $a = 3$.

Further reading

- **Ian Stewart, *Does God Play Dice?*** (Penguin). The classic popular account of chaos, including the logistic map.
- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). The standard undergraduate introduction; Chapter 10 is about one-dimensional maps like this one.
- **Marcus du Sautoy, *The Music of the Primes*** (Fourth Estate). Includes the story of the hunt for Mersenne primes.

2.9 Fractions That Close In on e

The question

- (i) Show that, for $n = 2, 3, 4, \dots$,

$$\frac{d^2}{dt^2} \left(t^n (1-t)^n \right) = nt^{n-2} (1-t)^{n-2} \left[(n-1) - 2(2n-1)t(1-t) \right].$$

- (ii) The sequence $T_0, T_1, \dots$ is defined by

$$T_n = \int_0^1 \frac{t^n (1-t)^n}{n!} e^t dt.$$

Show that, for $n \geq 2$,

$$T_n = T_{n-2} - 2(2n-1)T_{n-1}.$$

- (iii) Evaluate T_0 and T_1 and deduce that, for $n \geq 0$, T_n can be written in the form

$$T_n = a_n + b_n e,$$

where a_n and b_n are integers (which you should not attempt to evaluate).

- (iv) Show that $0 < T_n < \frac{e}{n!}$ for $n \geq 0$. Given that b_n is non-zero for all n , deduce that $\frac{-a_n}{b_n}$ tends to e as n tends to infinity.

The hook

How would you find good fractions approximating e ? You could truncate the series $1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$, but the fractions you get are not especially good for the size of their denominators. This question does something much cleverer. It builds a family of integrals, each of which is a small number of the form “integer + integer $\times e$ ”. If $a + be$ is tiny, then e is very close to $-a/b$, and the integrals hand us a sequence of excellent fractions: $\frac{19}{7}, \frac{193}{71}, \frac{2721}{1001}, \dots$

The same trick, integrals that are simultaneously small and of the form “integer combination of the number you care about”, is behind some of the deepest irrationality proofs in mathematics, including Hermite’s proof that e is transcendental and Apéry’s proof that $\zeta(3)$ is irrational.

Before you start

Look at the integrand $\frac{t^n(1-t)^n}{n!}e^t$ on $[0, 1]$.



Ask yourself What is the largest value of $t(1-t)$ on $[0, 1]$? What does that tell you about the size of T_n as n grows?

The maximum of $t(1-t)$ is $\frac{1}{4}$, at $t = \frac{1}{2}$. So the integrand is at most $\frac{e}{4^n n!}$, which shrinks extraordinarily fast (Figure 2.9.1). That smallness is half the story. The other half, that T_n has the special form $a_n + b_n e$, comes from integrating by parts.

Working it through

STAGE 1 Differentiating twice

Let $p(t) = t^n(1-t)^n$. By the product rule,

$$p'(t) = nt^{n-1}(1-t)^n - nt^n(1-t)^{n-1} = nt^{n-1}(1-t)^{n-1}(1-2t).$$



Ask yourself Differentiate again. You will get two terms; take out the common factor $nt^{n-2}(1-t)^{n-2}$. To match the required form, you need to rewrite $(1-2t)^2$ in terms of $t(1-t)$. How?

Differentiating $n[t(1-t)]^{n-1}(1-2t)$ by the product rule (note that $\frac{d}{dt}[t(1-t)] = 1-2t$):

$$\begin{aligned} p''(t) &= n(n-1)[t(1-t)]^{n-2}(1-2t)^2 - 2n[t(1-t)]^{n-1} \\ &= nt^{n-2}(1-t)^{n-2}[(n-1)(1-2t)^2 - 2t(1-t)]. \end{aligned}$$

Now $(1-2t)^2 = 1 - 4t + 4t^2 = 1 - 4t(1-t)$, so the bracket is $(n-1) - 4(n-1)t(1-t) - 2t(1-t) = (n-1) - 2(2n-1)t(1-t)$. ■

STAGE 2 Integrating by parts twice



Ask yourself $p(t)$ is zero at both $t = 0$ and $t = 1$. What about $p'(t)$, for $n \geq 2$? Why does that matter when you integrate $p(t)e^t$ by parts?

For $n \geq 2$, $p'(t)$ contains the factor $t^{n-1}(1-t)^{n-1}$, so it also vanishes at both ends. Integrating by parts twice, every boundary term is zero:

$$\int_0^1 p e^t dt = [p e^t]_0^1 - \int_0^1 p' e^t dt = -[p' e^t]_0^1 + \int_0^1 p'' e^t dt = \int_0^1 p'' e^t dt.$$

From Stage 1, $p''(t) = n(n-1)t^{n-2}(1-t)^{n-2} - 2n(2n-1)t^{n-1}(1-t)^{n-1}$. Divide by $n!$:

$$\frac{p''(t)}{n!} = \frac{t^{n-2}(1-t)^{n-2}}{(n-2)!} - 2(2n-1)\frac{t^{n-1}(1-t)^{n-1}}{(n-1)!}.$$

Multiplying by e^t and integrating gives $T_n = T_{n-2} - 2(2n-1)T_{n-1}$. ■

STAGE 3 The integer structure

$T_0 = \int_0^1 e^t dt = e - 1$. For $T_1 = \int_0^1 (t - t^2)e^t dt$, integration by parts gives $\int_0^1 te^t dt = 1$ and $\int_0^1 t^2 e^t dt = e - 2$, so $T_1 = 1 - (e - 2) = 3 - e$.



Ask yourself $T_0 = -1 + 1 \cdot e$ and $T_1 = 3 + (-1) \cdot e$. Why must every T_n have the form $a_n + b_n e$ with integer a_n, b_n ?

By strong induction: if $T_{n-2} = a_{n-2} + b_{n-2}e$ and $T_{n-1} = a_{n-1} + b_{n-1}e$ with integer coefficients, then

$$T_n = (a_{n-2} - 2(2n-1)a_{n-1}) + (b_{n-2} - 2(2n-1)b_{n-1})e,$$

and the new coefficients are integers too. ■

STAGE 4 Small, and therefore informative

For $0 < t < 1$, the integrand is positive, and $t^n(1-t)^n \leq 1$ and $e^t < e$. So

$$0 < T_n < \int_0^1 \frac{e}{n!} dt = \frac{e}{n!}.$$



Pause and try Divide $0 < a_n + b_n e < \frac{e}{n!}$ by $|b_n|$. What does that say about the distance between e and $-a_n/b_n$?

Dividing by $|b_n|$, which is a positive integer and so at least 1,

$$\left| e - \left(\frac{-a_n}{b_n} \right) \right| = \frac{|a_n + b_n e|}{|b_n|} < \frac{e}{n! |b_n|} \leq \frac{e}{n!}.$$

The right-hand side tends to 0, so $-a_n/b_n \rightarrow e$. ■

Here are the first few, computed from the recurrence:

n	2	3	4	5	6
$-a_n/b_n$	$\frac{19}{7}$	$\frac{193}{71}$	$\frac{2721}{1001}$	$\frac{49171}{18089}$	$\frac{1084483}{398959}$
error	4×10^{-3}	3×10^{-5}	1×10^{-7}	3×10^{-10}	5×10^{-13}

The fraction $\frac{2721}{1001}$ is correct to 6 decimal places with only a four-digit denominator.

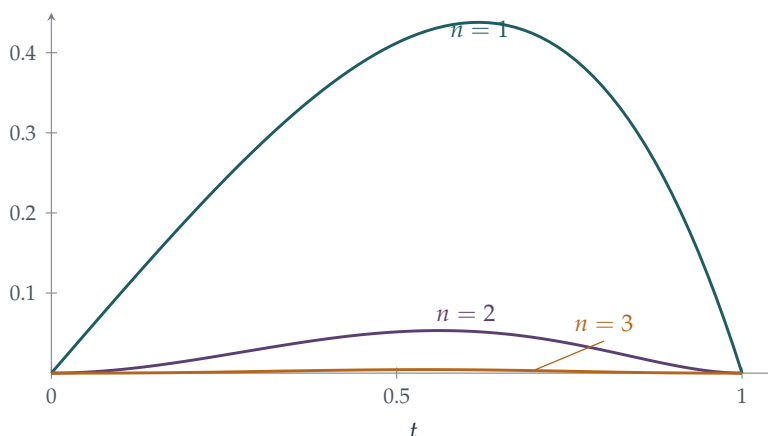


Figure 2.9.1 The integrands $\frac{t^n(1-t)^n}{n!}e^t$ for $n = 1, 2, 3$. Each is positive on $(0, 1)$, so each T_n is positive; and they shrink so quickly that by $n = 3$ the curve is barely visible. Positive but tiny is exactly what the argument needs.

Looking back

- **Integration by parts as a recurrence machine.** Because the polynomial and its derivative vanished at both ends, integrating by parts gave a clean recurrence with no boundary terms.
- **Structure and size together.** The recurrence showed T_n has the form $a_n + b_n e$; the bound showed it is small. Neither alone says anything about approximating e .
- **Don't compute what you don't need.** The question told us not to evaluate a_n and b_n , and the argument never needed them; it needed only that they were integers.

What's really going on

Another proof that e is irrational

The question stops just short of a famous theorem. Suppose $e = p/q$ with p, q positive integers. Then

$$q T_n = q a_n + b_n p$$

is an integer. But $0 < q T_n < \frac{q e}{n!}$, and for large n the right-hand side is less than 1. So $q T_n$ would be an integer strictly between 0 and 1, which is impossible. Hence e is irrational. This is the “trap an integer between 0 and 1” idea from Section 1.4, delivered by integrals instead of series. Integrals of exactly this kind, with polynomials like $t^n(1-t)^n$, were used by Charles Hermite in 1873 to prove that e is *transcendental*: it is not a root of any polynomial with integer coefficients.

Continued fractions

The fractions in the table are not just good; they are among the *best possible*. The continued fraction of e , found by Euler, has a beautiful pattern:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots],$$

and its convergents are $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \dots, \frac{2721}{1001}, \dots$. Our $-a_n/b_n$ are exactly every third convergent. No fraction with a smaller denominator gets closer to e . Euler's pattern was itself the first proof that e is irrational, since only rational numbers have finite continued fractions.

Apéry's miracle

In 1978 Roger Apéry announced a proof that $\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \dots$ is irrational, a question that had been open since Euler (see Section 2.3). His proof was so unexpected that the audience at first did not believe it. A year later Frits Beukers found a short version using integrals of the form

$$\int_0^1 \int_0^1 \frac{-\ln(xy)}{1-xy} P_n(x) P_n(y) \, dx \, dy,$$

where P_n is built from $\frac{d^n}{dx^n} (x^n(1-x)^n)$. These integrals are small, positive, and of the form “integer + integer $\times \zeta(3)$ ”, divided by a controlled denominator: exactly the strategy of this question. Whether $\zeta(5)$, or $e + \pi$, is irrational is still unknown.

The thinking skill

Small but not zero

If a quantity is known to be an integer combination of some numbers, and also known to be small but not zero, it carries information: about approximations, and sometimes about irrationality.

The pattern is “structure from algebra, size from analysis”. It recurs throughout number theory, and it is one of the places where analysis and algebra meet most fruitfully.

Try this next

- 1 Run the recurrence.** Compute a_n and b_n for $n = 2, 3, 4$ from the recurrence and check the fractions in the table. Show that the signs of b_n alternate.
- 2 A sharper bound.** Show that $0 < T_n < \frac{e}{4^n n!}$. How much does this improve the error estimate for $-a_n/b_n$?

- 3 The irrationality proof.** Write out the argument in the box above carefully. Where exactly do you use that $T_n > 0$?
- 4 A different weight.** **HARDER** Repeat the question with e^t replaced by e^{2t} (so $U_n = \int_0^1 \frac{t^n(1-t)^n}{n!} e^{2t} dt$). Find a recurrence for U_n , show that U_n is a combination of 1 and e^2 with rational coefficients, and investigate whether the argument still proves that e^2 is irrational. (Compare Section 1.4.)

Further reading

- **Ivan Niven, *Irrational Numbers*** (Mathematical Association of America). The classic short book on irrationality proofs, including Hermite's method.
- **Alfred van der Poorten, "A proof that Euler missed", *The Mathematical Intelligencer* 1** (1979). A lively account of Apéry's proof and its reception.
- **Julian Havil, *The Irrationals*** (Princeton University Press). A history of irrational numbers, written for a general mathematical reader.

What We Have Learned

We began with Zeno, and with three questions about infinite processes: what it means for them to have an answer, how to prove that they do, and how fast they get there. Nine STEP questions later, here is how far we have come.

1. What does it mean for an infinite process to have an answer?

In Section 2.1 we learned the language. A statement about the long-run behaviour of a sequence has the shape “there exists a point m such that for all $n \geq m \dots$ ”. Proving it means producing m ; disproving it means showing that every proposed m fails. The definition of a limit adds one more quantifier at the front, “for every tolerance $\varepsilon > 0$ ”, and with that, Zeno’s paradox dissolves: $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$ is within any tolerance you like of 1 from some point on, and that is all that “the sum is 1” means.

The language also guards against nonsense. The quick “let x equal the infinite expression” argument gave the right answer for the golden ratio in Section 2.2, but only because we could separately prove that the limit existed. The same argument applied to $1 - 1 + 1 - \dots$ gives an answer that means nothing.

2. How can we prove convergence without knowing the limit?

Most of the chapter’s sections used one of a handful of tools.

- **Monotone and bounded.** An increasing sequence that is bounded above must converge (Section 2.2, Section 2.6). The same applies to partial sums of positive terms (Section 2.3). At university you will see that this is really a statement about the real numbers themselves: it is the property of *completeness*.
- **Comparison.** A series of positive terms converges if it is smaller than one that converges, and diverges if it is bigger than one that diverges. Grouping terms into blocks made the comparison easy, and showed that the harmonic series diverges while Kempner’s thinned-out version converges (Section 2.3).
- **The squeeze.** Trap the quantity between two simpler ones with the same limit. This is how we proved $\sum 1/m^2 = \pi^2/6$ (Section 2.4) and evaluated the AGM integral rigorously (Section 2.6).
- **Exact formulas for finite stages.** In Section 2.5 and Section 2.8 we found exact expressions after n steps, by telescoping, and let $n \rightarrow \infty$ only at the very end.

Once a limit is known to exist, it can often be found by “letting $n \rightarrow \infty$ ” in a recurrence, as with $L = (2L + 1)/(L + 1)$ for the golden ratio. But the order matters: existence first, value second.

3. How fast?

Here the chapter held some surprises. Viète’s product (Section 2.5) gains about 0.6 of a decimal digit per factor. The simple recipe for $\sqrt{2}$ in Section 2.7 gains about 0.8 digits per step. Newton’s method and the AGM (Section 2.6) *double* the number of correct digits at each step: ten steps of Newton’s method give more than 700 digits of $\sqrt{2}$. The difference is the shape of the error recurrence: “new error $\approx$ constant $\times$ old error” (linear) against “new error $\approx$ constant $\times$ (old error)²” (quadratic). And in Section 2.8 we saw the opposite extreme: starting just above 2, the map $x \mapsto x^2 - 2$ pushes numbers away faster than exponentially, while inside $[-2, 2]$ it amplifies tiny errors until all predictability is lost.

Speed also matters for theory. The integrals in Section 2.9 produce fractions approaching e so quickly that they prove e is irrational: no fraction p/q could be approached that fast by other fractions.

4. A toolkit for analysis

Method	Where you used it
Producing and negating quantifiers	“Eventually” (2.1)
Monotone bounded sequences	Golden ratio (2.2); AGM (2.6)
Comparison and condensation	Harmonic and Kempner series (2.3)
Squeezing	$\pi^2/6$ (2.4); the AGM integral (2.6)
Symmetric functions of roots	Sum of $\cot^2$ values (2.4)
Telescoping sums and products	Viète (2.5); $x^2 - 2$ (2.8)
Invariants	The AGM integral (2.6)
Error recurrences	Linear and quadratic convergence (2.7)
Changing variables to simplify a map	$a = t + 1/t$, $a = 2 \cos \theta$ (2.8)
Small but non-zero integer combinations	Approximating e (2.9)

Notice how many of these tools came with a “theorem you may use” attached. STEP often hands you the monotone convergence theorem, or tells you to assume that limits exist. At university you will prove those theorems from the axioms of the real numbers, and you will be expected to notice for yourself when an argument quietly assumes a limit exists. When you read a solution, it is a very good habit to ask: *where exactly was a limit taken, and was it justified?*

Where next

The ideas of this chapter run through the rest of the book.

- Fixed points and cobweb diagrams return in Section 8.7, where a chaotic map turns out to have an exact solution.
- The harmonic numbers $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ appear in probability in Section 9.11 and in mechanics in Section 11.12.

- ▶ Another appearance of π in a limit, from probability, is in Section 12.23.
- ▶ The careful attitude to limits is the theme of Chapter 3, which opens with Section 3.1: arguments that look right but aren't.

If you want to go further before university

- ▶ **Lara Alcock, *How to Think About Analysis*:** what first-year analysis is like, and how to cope with it.
- ▶ **David Bressoud, *A Radical Approach to Real Analysis*:** analysis taught through its history, starting from the puzzles that forced mathematicians to be careful.
- ▶ **Tom Körner, *Calculus for the Ambitious*** (Cambridge University Press): written by a Cambridge analyst for exactly the readers of this book.
- ▶ **Julian Havil, *Gamma*** and **William Dunham, *Euler: The Master of Us All*:** the stories behind many of the series in this chapter.

Zeno thought infinity made motion impossible. Mathematicians took two thousand years to answer him, and the answer turned out to be one of the most powerful ideas ever found: a way of making precise statements about processes that never end. You now know the first few words of that language. Analysis is where you will learn to speak it fluently.